DETERMINATION OF CURRENT ASSETS DEMAND

Paulo Jorge Miranda PANARELLA[*]

Abstract. *This paper derives current assets demand and optimal production, inventory and credit policies, simultaneously. Methodologically, this paper is a theoretical essay hinging on mathematical modeling and dynamic optimization, building on Accounting and Finance conceptions organized in the setting of neoclassical microeconomics. The originality of this paper is the unique combination of Accounting, Finance, neoclassical microeconomics and optimal control theory to derive the results which become more precise when taken mathematically explicit Accounting and Finance conceptions: (i) inventory flow measurement criteria, (ii) direct and indirect taxes, (iii) discounted cash flow as objective function instead of profit and (iv) valuation principle. The results derived are: optimal production trajectory, optimal raw material and finished goods inventory trajectories, optimal credit period, optimal current assets composition and amount demanded and a set of equations that extends neoclassical theory from static into a dynamic which explicitly deals with production, inventory and credit policies altogether in line with Accounting and Finance valuation principles.*

Keywords: *Working capital, current assets, economic dynamics, neoclassical microeconomics.*

1. Introduction

Current assets take part in investment decision, accounting for more than fifty per cent of total investment and requiring at least half of managers time [1]. Current assets support operational strategy, for example, growing sales policy to enlarge market share. Accordingly, firms often make use of credit to enlarge sales, however excess credit lowers liquidity position, thus exposing insolvency and bankruptcy for overtrading.

Financial literature has long addressed the issue of investing, providing knowledge and tools for appropriately managing long term investments (e.g. fixed assets). Although finance literature has also dealt with short term issues such as current assets determination and

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[*] Independent, Brazil, paulo.panarella@gmail.com.

management, awareness is still fragmented, preventing complete assessment [2] and [3]. Thus a unified framework is required to integrate knowledge. This working paper addresses this issue and contributes to integrate the different parts of short term finance in a consistent manner to assess current assets determination. In this regard, the research question this working paper seeks to answer is: *what is the initial amount to be invested in current assets given an operating context?*

Related to our research question, the prime objective of this working paper is to determine the exact amount to invest in current assets contingent on operating context. The secondary objective is to access how investment in current assets may be adjusted during fiscal year.

In addition to this introductory section, this working paper is organized as follows. Section 2 describes methodology. Section 3 presents a review of accounting and finance concepts related to working capital. Section 4 shows model description. Section 5 deals with model solution. Section 6 treats operating cash flow determination. Section 7 presents working capital determination. Section 8 concludes this paper.

2. Methodology

This working paper consists of a theoretical essay hinging on mathematical modeling. The purpose is to derive a solution from the model that describes the relation between current assets, sales forecast and costs. Based on the solution obtained, it will be possible to understand how credit sales, production and inventory policies determine the demand for current assets, specifically, total amount and how it splits into different accounts.

Our model will be based on the following notions and theories: (i) Accounting concepts, (ii) Finance conceptions, namely valuation, and (iii) Neoclassical Microeconomics (maximizing behavior of economic entities and cost functions).

3. Review of Accounting and Finance

This section is dedicated for reviewing accounting concepts and finance conceptions that are useful for our research objectives.

3.1. Accounting Statements

In what follows, we review accounting statements: balance sheet, income statement and cash flow statement. All the important concepts are condensed in these three statements.

3.1.1. Balance sheet

The objective of balance sheet is to demonstrate financial position at each date $t = 1, 2, \ldots, n$. Financial position will be interpreted as state variable.

Assets		**Liabilities and Equity**	
Cash	M_t (money demand)	Suppliers	RMP_t (RM Purchase)
Accounts Receivable	s_t (term sales)	Salaries and Rents	SR_t
Finished Goods Inventory	FGI_t (money units), y_t (physical units)	Taxes	Tax_t
Raw Material Inventory	RMI_t (money units), x_t (physical units)	Bonds (short or long term)	SB_t, LB_t
Fixed Assets	FA_t	Equity	E_t
Total Assets (Investment)	$M_t + s_t + FGI_t + RMI_t + FA_t$	Total Liabilities plus Equity (Funding)	$RMA_t + SR_t + Tax_t + SB_t + LB_t + E_t$

Assets correspond to investment made by the firm in acquiring both economic resources and rights. They split in current assets (cash, accounts receivable, finished goods inventory and raw material inventory) and fixed assets (machines, equipments, facilities etc.).

Current assets respond directly to daily operations (purchase, production, inventory, sales and cash collecting). Current assets are cyclical in the following: cash converts into raw material inventory after purchase, which converts into finished goods inventory after production, then accounts receivable after finished goods sales and, finally, return to cash after client payments. Basically, current assets are demanded for transaction and liquidity purposes. Current assets determination is the core issue of this working paper.

Fixed assets correspond to installed capacity to support operations, thus they are not intended to be sold. They vary with sales only in the long run when firm has enough time to adjust its capacity according to long term profitability (which depends on strategic decisions taking into account future scenarios, business life cycle itself etc.). Thus, in the short run, fixed assets don't vary with sales; in spite of this, loose value in the form of depreciation. In this working paper we will not model the decision to invest in fixed assets. We will take them as initial conditions. However we suggest the simultaneous determination of current and fixed assets for future research.

Liabilities refer to obligations, i. e., third party claims or rights in future cash generated by the firm. They reflect both the financing decision of the firm and market credit conditions (and institutions). Liabilities split into current liabilities (short term obligations: bank loans, bonds, suppliers, salaries and rents, taxes) and noncurrent liabilities (long term obligations: bank loans, bonds etc.). In this paper we will not address financing decision taking, only automatic short term obligations, i. e., credit that is taken as usual and must be paid in the following month (suppliers, salaries and rents and taxes). Combining the simultaneous determination of current and fixed assets with financing decision taken (determination of an adequate mix of obligations and equity) is out of our purpose, but is left as suggestion for future research.

Equity represents own capital, i. e., investment made in the firm by its owners in the form of quotas or stocks (depending on the firm being private or public). Those owners have right only to a residual income (cash or income that rests after all obligations have been paid); therefore, they concentrate more risk.

3.1.1.1. Balance sheet identities

Left side of balance sheet (total investments) equals its right side (total funding), thus the following accounting identity holds:

$$A = B + E.$$

Letter A denotes total assets, B total debts (obligations) and E means equity. We're denoting total debts with caps letter B (Bonds) not to confuse with D which will represent depreciation.

The above identity is known as *Patrimonial Equation*. As all accounting identities, Patrimonial Equation may be interpreted as a *conservation law* (like mass or energy). Taking all balance sheet accounts together, for every time date (t = 1, 2, …, n), Patrimonial Equation holds and is rewritten as:

$$M_t + s_t + FGI_t + RMI_t + FA_t = RMP_t + SR_t + Tax_t + SB_t + LB_t + E_t.$$

Current Assets (*CA*), also known as *Gross Working Capital* (*GWC*), is the next sum:

$$CA_t = M_t + s_t + FGI_t + RMI_t.$$

Working Capital (*WC*), also known as *Net Current Asset* (*NCA*), is, by definition, the following difference:

$$WC_t = CA_t - CB_t.$$

Where *CB* (*Current Liabilities*) is the sum of all obligations (debts) due to current fiscal year: S*hort Term Bonds* (SB_t) + S*uppliers* (RMP_t) +

S*alaries and Rents* (SR_t) + *Taxes* (Tax_t). We use B (the initial letter of Bond) to represent liabilities because we´re reserving L to denote labor intensity, later on. Note also that we're using RMP_t for Suppliers (meaning obligation due to Raw Material Purchase) not to confuse with Term Sales (Accounts Receivable) denoted by s_t.

Because Patrimonial Equation holds, the following accounting identity is also true:

$$WC_t = LB_t + E_t - FA_t.$$

The above identity allows us to interpret WC as follows:

(i) $WC > 0$ means the firm has enough resources (due to long term: bond plus equity) to finance capacity (fixed assets) and, at least, part of its operations needs (current assets). In fact, firm has working capital. This case corresponds to a strong liquidity position or a conservative strategy, but lower return. However, as we are not dealing with financing decision taking, $LB_t = 0$, i.e., we are supposing the funding is 100% equity. Consequently, working capital reduces to *Own Working Capital* (*OWC*), $WC_t = OWC_t$, where $OWC_t = E_t - FA_t$.

(ii) $WC < 0$ means the firm has insufficient resources to finance capacity, thus requires short term obligations to partially finance them as well as the whole of operations needs (current assets). This case corresponds to a weak liquidity position or an aggressive strategy (higher return, but with more risk).

(iii) $W = 0$ means long term resources completely finances capacity while short term obligations is financing all operation needs and cash. This case corresponds to financial equilibrium where maturity of obligations matches maturity of investments.

Defining *Operating Current Assets* (*OCA*) as $OCA_t = s_t + FGI_t + RMI_t$ and *Operating Current Liabilities* (*OCB*) as $OCB_t = RMP_t + SR_t + Tax_t$ yields the definition of *Working Capital Requirement* (*WCR*) as follows:

$$WCR_t = OCA_t - OCB_t\ .$$

When $WCR > 0$ firm faces insufficiency of operating credit (such as suppliers), thus has a need to finance operations by other sources. When $WCR < 0$ firm completely finances operations with operating credit and still has a surplus that goes either to cash or short term applications. When $WCR = 0$, operating credit perfectly matches operations needs.

Finally, taking the difference between WC and WCR, we have the next identity:

$$WC_t - WCR_t = M_t - SB_t\ .$$

When $WC > WCR$, firm has a surplus financing in cash: long term resources completely finance both operations and capacity, therefore what rests from them goes straight to cash, a bold liquidity position. When $WC < WCR$, firm faces insufficiency of long term resources to finance both operations and cash, thus needs extra money coming from short term loans or bonds, a poor liquidity position (firm risks insolvency). When $WC = WCR$, long term resources completely finance both operations and capacity, but cash has to be totally financed by short term loans, a more neutral liquidity position followed by lower risk of insolvency.

3.1.2. Income statement

The objective of income statement is to demonstrate income generation a long fiscal year. The bottom line (Net Profit or Net Income) is a measurement of economic performance or wealth.

Items of Income Generation	$t = 1$	$t = 2$	…	$t = \mathrm{n}$	Total
Net Revenue	$\boldsymbol{NR_1}$	$\boldsymbol{NR_2}$	…	$\boldsymbol{NR_\mathrm{n}}$	$\boldsymbol{\Sigma NR_\mathrm{t}}$
(–) Cost of Goods Sold	CGS_1	CGS_2	…	CGS_n	ΣCGS_t
(–) Operating Expenses	OE_1	OE_2	…	OE_n	ΣOE_t
Operating Profit	$\boldsymbol{OP_1}$	$\boldsymbol{OP_2}$	…	$\boldsymbol{OP_\mathrm{n}}$	$\boldsymbol{\Sigma OP_\mathrm{t}}$
(–) Interest Expenses (Interests)	IE_1	IE_2	…	IE_n	ΣIE_t
Net Operating Profit (Income)	$\boldsymbol{NOP_1}$	$\boldsymbol{NOP_2}$	…	$\boldsymbol{NOP_\mathrm{n}}$	$\boldsymbol{\Sigma NOP_\mathrm{t}}$
(–) Income Tax	IT_1	IT_2	…	IT_n	ΣIT_t
Net Profit (Net Income)	$\boldsymbol{NP_1}$	$\boldsymbol{NP_2}$	…	$\boldsymbol{NP_\mathrm{n}}$	$\boldsymbol{\Sigma NP_\mathrm{t}}$

Thus, the most important accounting identity here is net profit, which is defined as:

$$NP_t = (1 - \beta) \cdot [(1 - \alpha)R_t - CGS_t - OE_t - IE_t].$$

In the above identity, the symbol R_t refers to total revenue; α, β denote, respectively, tax revenue and income tax aliquots. Note that we have:

$$NR_t = (1 - \alpha)R_t \text{ and } NOP_t = NR_t - CGS_t - OE_t - IE_t.$$

As we're supposing 100% equity funding, interest expenses are null, therefore net profit reduces to:

$$NP_t = (1 - \beta) \cdot [(1 - \alpha)R_t - CGS_t - OE_t].$$

NP is also known as accounting profit. We elaborate more on profit concepts in section 3.2

3.1.3. Cash flow statement

The objective of cash flow statement is to demonstrate cash generation (needs) during fiscal year. As we are only concerned with operations (neither investment nor financing), cash flow statement reduces to operations items of cash.

There are two ways to demonstrate cash: direct method and indirect method. Direct method relates cash inflow and cash outflow directly as follows.

Direct Method	$t = 1$	…	$t = 1+\tau$	…	$t = n+\tau$	Total
Client Receipts (Revenue)	$\boldsymbol{s_0}$	**…**	$\boldsymbol{R_1}$	**…**	$\boldsymbol{R_n}$	$\boldsymbol{\Sigma R_t}$
(–) Raw Material Purchase	RMP_0	…	RMP_1	…	RMP_n	ΣRMP_n
(–) Salaries and Rents	SR_0	…	SR_1	…	SR_n	ΣSR_t
(–) Taxes	Tax_0	…	Tax_1	…	Tax_n	ΣTax_t
Operating Cash Flow	$\boldsymbol{OCF_0}$	**…**	$\boldsymbol{OCF_1}$	**…**	$\boldsymbol{OCF_n}$	$\boldsymbol{\Sigma OCF_t}$

Indirect method transforms net profit into operating cash flow as follows, see [4] and [5]. The objective of this transformation is to inform how profit is generating cash.

Indirect Method	$t = 1$	…	$t = 1+\tau$	…	$t = n+\tau$	Total
Net Profit	$\boldsymbol{NP_1}$	**…**	$\boldsymbol{NP_i}$	**…**	$\boldsymbol{NP_j}$	$\boldsymbol{\Sigma NP_t}$
(+) Depreciation	D_1	…	D_i	…	D_j	ΣD_t
(–) Working Capital Requirement Variation	ΔWCR_1	…	ΔWCR_i	…	ΔWCR_j	$\Sigma\Delta WCR_t$
Operating Cash Flow	$\boldsymbol{OCF_0}$	**…**	$\boldsymbol{OCF_1}$	**…**	$\boldsymbol{OCF_n}$	$\boldsymbol{\Sigma OCF_t}$

The advantage of indirect method is to assess how net profit converts into operating cash flow through working capital requirements variation. As this variation is directly explained by changes in term sales and inventory policies, indirect method allows immediate awareness of the impact of those policies changes on cash generation (or cash needs). In other words, cash dynamics is more explicit in indirect method.

3.2. Accounting Profit, Economic Profit, Free Cash Flow to Firm and Valuation Principle

In this section, we discuss accounting profit, economic profit and free cash flow to firm. The aim of this discussion is to establish a foundation to compute a performance function to evaluate firm wealth generating.

Accounting profit (net profit in section 3.1.2) considers only accounting costs: depreciation plus explicit costs, i. e., costs that generate disbursements due to usage and rental of third party resources.

Accounting costs as well as other accounting concepts were designed for external financial reporting [6], however they may lack relevance to inform mangers about economic performance [7]. First, accounting costs lack opportunity cost, i. e., forgiven benefits due to alternative uses of own capital. Second, accounting costs include sunk costs, i.e., costs that are unrecoverable (truly losses) even if the firm regrets and tries to revert decision. As such, sunk costs don't belong to future decisions, thus they must be excluded of cost-benefit analysis.

Economic profit considers some accounting costs, includes opportunity costs and excludes sunk costs (part of accounting costs that become sunk if decision would be reverted). The next identity [8] sums up the relation between accounting profit (NP_t) and economic profit (π_t):

$$\pi_t = NP_t - OpC_t + SuC_t.$$

In the above identity, opportunity costs are denoted by OpC_t while sunk costs are represented by SuC_t.

Free cash flow to firm (*FCF*) refers to all cash generated by operations and that are free to be distributed into creditors and firm owners (equity), after purchases, taxes and salaries (and rents) have been paid. As we consider equity as the sole source of funding, cash outflows due to creditors vanish, thus free cash flow to firm reduces to operating cash flow, i. e., $FCF = OCF$.

Then, the relation between accounting profit (*NP*) and operating cash flow (*OCF*) is summed up by the next identity [1], [4], [5] and [9]:

$$OCF_t = NP_t + D_t - \Delta WCR_t.$$

Taking into account the previous discussion, there are three alternative performance measures (value functional) to evaluate wealth generating:

A) Performance measure based on accounting profit (*NP*)

$V[NP] = \int_0^T e^{-rt} NP(t)dt$ (continuous time), $V[NP] = \sum_{t=1}^{n}(1 + r)^{-t} NP_t$ (discrete time);

B) Performance measure based on economic profit (π)

$V[\pi] = \int_0^T e^{-rt} \pi(t)dt$ (continuous time), $V[\pi] = \sum_{t=1}^{n}(1 + r)^{-t} \pi_t$ (discrete time);

C) Performance measure based on operating cash flow (*OCF*)

$V[OCF] = \int_0^T e^{-rt} OCF(t)dt$ (continuous time), $V[OCF] = \sum_{t=1}^{n}(1 + r)^{-t} OCF_t$ (discrete time).

According to valuation principle, the value of an asset is directly determined by its associated free cash flow stream (see [1], [4], [5], [9] and [10]). Thus, to be consistent to valuation principle, we choose as value functional alternative *C*, which is based on free cash flow (operating cash flow).

4. Model Description

The model is based on a single commodity firm with stable seasonable demand and known costs, in a deterministic setting and continuous time. The complete description is as follows:

4.1. Forecasted Demand

Forecasted demand is a wave which increases constantly with credit period τ (terms of sale)

$$q(t, \tau) = q_0 + B\tau + A\cos(\varphi_0 + \omega t) = 10 + \frac{1}{6}\tau + 10\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) .$$

The above expression is valid if τ belongs to {30,60,90} and $t > 0$. Otherwise, $q = 0$ for all sales are for credit and starts just after date 0. Credit period is the only term of credit policy in this paper (for a more complete view of credit policy see [4]). Besides, credit period is the same for all transactions and price is constant: $p = 50$. Revenue and income taxes aliquots are $\alpha = 10\%$ and $\beta = 40\%$, respectively.

4.2. Forecasted Costs

Forecasted costs are a total flow ($/day) including production costs as well as operating expenditures:

$$C = C(u, y, x, s. q) = C^u(u) + C^y(y) + C^x(x) + C^s(s) + C^q(q) .$$

In the above expression, we're supposing total costs *C* are separable into five independent cost drivers (*u*,*y*,*x*,*s*,*q*) which means the absence of any interaction among them. The question of costs separability and aggregation into independent cost pools may be seen in [11], [12] and [13].

Production cost is dependent on production flow $u \geq 0$ (output unit per day), described by a quadratic function with decreasing returns (as standard Microeconomics):

$$C^u(u) = a_u u^2 + b_u u + c_0^u = \frac{1}{4}u^2 + 8u + 50.$$

Finished goods inventory expenditure is dependent on finished goods inventory $y \geq 0$ (output unit position), described by a quadratic function with both increasing and decreasing returns:

$$C^y(y) = a_y y^2 - b_y y + c_0^y = y^2 - 10y + 54\ .$$

Raw material expenditure is dependent on raw material inventory $x \geq 0$ (raw material unit position) described by a quadratic function with both increasing and decreasing returns:

$$C^x(x) = a_x x^2 - b_x x + c_0^x = x^2 - 8x + 65\ .$$

Raw material price is constant: $w = 5$ (\$/ raw material unit)

Investment in client expenditure is dependent on investment in client $s \geq 0$ (dollar position of accounts receivable), described by a quadratic function with both increasing and decreasing returns:

$$C^s(s) = a_s s^2 - b_s s + c_0^s = \frac{1}{1250}\left(\frac{1}{25000}s^2 - \frac{2}{25}s + 55\right).$$

The quadratic forms of functions C^y, C^x, C^s are suggested by [4]. These quadratic forms are a way to represent the trade-off between the benefits and costs of holding inventory (and effectuating sales for credit). The decreasing part of the function means net benefits of holding inventory (term sales) are positive while the increasing part means net benefits are negative. However at a specific point, when net benefit is null (costs and benefits are offset), total cost achieve its minimum. Although it is tentative to set C^y, C^x, C^s in their respective minimum, we will show this choice is not always the case.

Sales expenditure completes our cost description. It is dependent on sales unit $q \geq 0$ (output unit per day), described by an affine function:

$$C^q(q) = b_q q + c_0^q = 2q + 25\ .$$

Note that all cost functions described have a variable and fixed component structure. This means they represent a short run setting because at least one factor cost is fixed. This notion is consistent with the idea of a firm operating during a fiscal year ($T = 360$ days).

Another point needs attention. We consider raw material and direct production labor as perfect complements. So, the relation between raw material usage v and labor intensity L becomes fixed, i. e., $L/v = a$, for some $a > 0$. Thus, labor intensity becomes linearly dependent on raw material usage: $L = L(v) = av$. This reduces total variable production cost to be dependent on a single independent variable v (since capital K is fixed):

$$VC^u = wv.$$

Supposing raw material price $w > 0$ constant (w = 5 \$/raw material unit), we restate:

$$v = \frac{VC^u}{w} = \frac{a_u u^2 + b_u u}{w} = \frac{\frac{1}{4}u^2 + 8u}{5} = \frac{1}{20}u^2 + \frac{8}{5}u\ .$$

In fact, supposing a twice differentiable production function $u = \mathrm{f}(v) = f(L(v)) = F(v,L(v),K)$, taking capital intensity K as fixed and marginal product $\partial F/\partial L > 0$, function f becomes invertible, thus we may put $v = g(u)$, where $g = f^{1} = VC^u/w$ is raw material requirement function (for each level of production). Conversely, $u = f(v)$ is production level for each raw material usage (requirement).

4.3. Conservation laws upon stock and flows

A conservation law is an accounting identity [13]. The first relates finished goods inventory (y), production flow (u) and sales unit flow (q) in what follows:

$$\dot{y} = u - q.$$

At each instant (time date t), the change in finished goods inventory is the difference between production flow (entry output unit per time) and quantity sales (exit output unit per time).

The second relates raw material inventory (x), raw material purchase unit requirement (z) and production (u) as this:

$$\dot{x} = z - \frac{VC^u(u)}{w}\ .$$

At each instant (time date t), the change in raw material inventory is the difference between raw material purchase unit requirement (entry raw material unit per time) and raw material usage as a function of production level (exit raw material unit per time).

4.4. Firm objective function

The firm aims to maximize value from free cash flow perspective (operating cash flow as discussed in item 3.2):

$$\max \int_0^{T+\tau} e^{-rt} OCF(t)\, dt = \int_0^{T+\tau} e^{-rt} \left[NP(t) + D(t) - \frac{d}{dt} WCR(t) \right] dt\,.$$

For analytical simplicity, we will set $r = 0$ (null or absent opportunity cost). In fact, if $r > 0$, the optimal value of term credit τ shortens, what doesn't add nor subtract any information. Besides, total variation of working capital requirement vanishes because integrating its time derivative from date 0 to date $T + \tau$ refers to a complete operating cycle after which all operating current accounts are converted into cash, provided $WCR(0) = 0$. As depreciation and income tax are constant, the maximization problem is equivalent to the following statement.

Problem 1: Determination of optimal policies of production, finished goods inventory and raw material inventory.

$$\max \int_0^{T=360} \{(1-\alpha) R\big(q(t,\tau)\big) - [C^u\big(u(t)\big) + C^y\big(y(t)\big) + C^x\big(x(t)\big) + C^s\big(s(t)\big) + C^q(q(t,\tau))]\}\, dt\,.$$

Subject to

$$\dot{y} = u - q$$

$$\dot{x} = z - \frac{VC^u(u)}{w}\,.$$

Initial and terminal conditions: $y_0, x_0, y_T, x_T = 0$.

5. Model Solution

Writing down the Hamiltonian attached to problem 1, we have:

$$H(y, x, u, z, \lambda, \mu) = (1-\alpha) R(q) - [C^u(u) + C^y(y) + C^x(x) + C^s(s) + C^q(q)]\,.$$

Using Pontryagin Maximum Principle (PMP), First Order Necessary Conditions (FONC) are:

$$0 = \frac{\partial H}{\partial u} = -MgC^u(u) + \lambda - \mu \frac{MgC^u(u)}{w} \quad (1)$$

$$0 = \frac{\partial H}{\partial z} = \mu \quad (2)$$

$$\dot{y} = \frac{\partial H}{\partial \lambda} = u - q \quad (3)$$

$$\dot{x} = \frac{\partial H}{\partial \mu} = z - \frac{VC^u(u)}{w} \quad (4)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} = MgC^y(y) \tag{5}$$

$$\dot{\mu} = -\frac{\partial H}{\partial x} = MgC^x(x). \tag{6}$$

Claim 1. Conditions (1) to (6) are also sufficient for a global maximum since the integrand is negative concave.

Claim 2. In the above conditions and the rest of this paper, marginal function is, by definition, the first derivative with respect to independent variable. Thus, we have:

a) $MgC^u(u) \overset{\text{def}}{\equiv} \frac{d}{du}C^u(u) = 2a_u u + b_u$ for marginal cost of production;

b) $MgC^y(y) \overset{\text{def}}{\equiv} \frac{d}{dy}C^y(y) = 2a_y y - b_y$ for marginal cost of finished goods inventory;

c) $MgC^x(x) \overset{\text{def}}{\equiv} \frac{d}{dx}C^x(x) = 2a_x x - b_x$ for marginal cost of raw material inventory.

Claim 3. Since from (1) and (2), $\lambda = MgC^u$ and $\mu = 0$, conditions (5) and (6) reduce to:

$$MgC^y(y) = \frac{d}{dt}MgC^u(u) \tag{5'}$$

$$MgC^x(x) = 0. \tag{6'}$$

Equation (5') states that the cost of the last unit that goes to finished goods inventory equals the cost change of the last unit of production. This condition is not commonly referred to in Microeconomics literature. However, it explains explicitly a stop condition for finished goods inventory which leads to maximum firm value. Besides, condition (5') can be easily used in practice (by managers) to control inventory and even to test if minimum inventory polices (advocated by business fads) really help to reach maximum firm value. Therefore, instead of using equation (3) (conservation law) together with an arbitrary percentage to define inventory policy (a very usual practice in business, see [17]), it could be used equation (3) with equation (5') to establish inventory policy that leads to maximum value.

The same discussion applies to equation (6'). However, there is a structural difference: the change in marginal cost of raw material is null (the term on the right side of equality sign) because marginal cost of raw material is constant, according to equation (2). This occurs due to the absence of control variable *z* (raw material purchase unit requirement) attached to state variable x (raw material inventory) as an explicit argument in objective function.

Solution of FONC. Solving the system (1) to (6), we find:

(i) *Optimal production policy:*

$$u(t,\tau) = \left(q_0 + B\tau - \frac{b_y}{2a_y}\sqrt{\frac{a_y}{a_u}}\frac{\cosh\sqrt{\frac{a_y}{a_u}}t}{\cosh\sqrt{\frac{a_y}{a_u}}T} \right) + A\cos(\varphi_0 + \omega t), \text{ or}$$

$$u(t,\tau) = \left(10 + \frac{1}{6}\tau - \frac{10\cosh 2t}{\sinh 720}\right) + 10cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) .$$

If t belongs to]0,360] and τ belongs to {30,60,90}; else, $u = 0$.

Note that the optimal production policy is a continuous wave that oscillates with respect to a diminishing quantity that jumps up with credit term sales *τ*. The oscillation and its reference point (which increases with τ) come from forecasted demand while the diminishing effect comes from the optimal finished goods inventory policy (solution of stop condition 5 or 5').

(ii) *Optimal finished goods inventory policy:*

$$y(t) = \frac{b_y}{2a_y}\left(1 - \frac{sinh\sqrt{\frac{a_y}{a_u}}t}{sinh\sqrt{\frac{a_y}{a_u}}T} \right), \text{ or } y(t) = 5\left(1 - \frac{sinh 2t}{sinh 720}\right).$$

If t belongs to]0,360]; else, $y = 0$.

Observe that the optimal finished goods inventory policy has as its supreme value the static global minimum point of cost function $C^y(y)$, the parameter $b_y/2a_y = 5$. The trajectory decreases very slowly, remaining quite close to that static global minimum almost all the way. However, at a certain time date t (as terminal date T is approached), the trajectory decreases fast until reaches zero at terminal date T. Although this changing behavior is profound, trajectory is continuous in the interval]0,360] since it is differentiable (solution of a differential equation).

(iii) *Optimal raw material policy:*

$$x(t) = \frac{b_x}{2a_x} = 4 .$$

If t belongs to]0,360]; else, $x = 0$.

Repair that the optimal raw material inventory policy is constant all the way in]0,360] at the static global minimum point of cost function $C^x(x)$. This characteristic comes from the fact that the condition

$MgC^x(x) = 0$ is optimal from dynamic perspective either for raw material purchase unit requirement (z), the controller of raw material inventory (x), is absent as an explicit argument in objective function (as we stated in claim 3). If this condition is violated, as is in the case of finished goods inventory optimality condition (see equation 5'); then, the optimal inventory policy may diverge from the static global minimum.

5.1. Determination of Investment in Client

We need to find an expression relating investment in client (s) and credit period (τ). To do so, we recall that the change in s increases with credit sales (total revenue at time t) and decreases with receipts (total revenue at time $t - \tau$): $\dot{s}(t) = R\big(q(t,\tau)\big) - R\big(q(t,t-\tau)\big)$, t belongs to $[0,T]$, $T = 360$, τ belongs to $\{30,60,90\}$.

Observe that $R(q(t,t-\tau)) = 0$, for all $t \leq \tau$, since no receipt is done before credit period. Supposing initial condition is $s_0 = 0$, investment in client is determined by the following development:

$$s(t,\tau) = \int_0^t R\big(q(\xi,\tau)\big) - R\big(q(\xi,\xi-\tau)\big)d\xi$$

$$s(t,\tau) = \int_{t-\tau}^t R\big(q(\xi,\tau)\big)\, d\xi$$

$$s(t,\tau) = \int_{t-\tau}^t p[q_0 + B\tau + A\cos(\varphi_0 + \omega\xi)]\, d\xi$$

$$s(t,\tau) = p\left[(q_0 + B\tau)\xi + \frac{A}{\omega}\sin(\varphi_0 + \omega\xi)\right]_{t-\tau}^t$$

$$s(t,\tau) = p\left\{(q_0 + B\tau)\tau + \frac{2A}{\omega}\sin\left(\frac{\omega\tau}{2}\right)\cos\left[(\varphi_0 + \omega t) - \frac{\omega\tau}{2}\right]\right\}.$$

Now, taking into account $p = 50$, $q_0 = 10$, $B = 1/6$, $A = 10$, $\varphi_0 = -\pi/2$ and $\omega = \pi/15$, yields:

$$s(t,\tau) = 500\tau + \frac{25}{3}\tau^2 + \frac{1500}{\pi}\sin\left(\frac{\pi}{30}\tau\right)\cos\left[\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) - \frac{\pi}{30}\tau\right].$$

Since $\sin 3\pi = \sin 2\pi = \sin \pi = 0$ if τ belongs to $\{30,60,90\}$, for each t in $]0,360]$, the expression of $s(t,\tau)$ reduces to:

$$s(t,\tau) = 500\tau + \frac{25}{3}\tau^2 = \begin{cases} 22500, \tau = 30 \\ 60000, \tau = 60 \\ 112500, \tau = 90 \end{cases}.$$

Note that the possible values of $s(t,\tau)$ are higher than the static global minimum of $C^s(s)$ which is $s^* = 1000$. Now, with the expression of investment in client in hand, we are able to find optimal credit period.

5.2. Determination of Credit Period

Substituting forecasted demand $q(t,\tau)$, optimal policies of production $u(t,\tau)$, finished goods inventory $y(t)$ and raw material inventory $x(t)$ together with the expression of investment in client $s(t,\tau)$ in the integral of problem 1, optimal credit period is found solving problem 2:

Problem 2. Determination of credit period

$$\max_\tau \int_0^{360} \{(1-\alpha)R(q(t,\tau)) - [C^u(u(t,\tau)) + C^y(y(t)) + C^x(x(t)) + C^s(s(t,\tau)) + C^q(q(t,\tau))]\}\,dt.$$

Computing the definite integral, we arrive at the following expression:

$$J(\tau) = -\frac{1}{1250}\tau^4 - \frac{12}{125}\tau^3 - \frac{1297}{250}\tau^2 + \frac{453581}{300}\tau + \frac{987354}{25}\ .$$

Recalling that τ belongs to $\{30,60,90\}$, we have:

$$J(\tau) = \begin{cases} 76943 \text{ } if \text{ } \tau = 30 \\ 80430 \text{ } if \text{ } \tau = 60 \\ 11074 \text{ } if \text{ } \tau = 90 \end{cases}.$$

So, the maximum value ($J_{\max} = 80430$) occurs when credit period is $\tau^* = 60$. Therefore, the optimal credit policy is $\tau^* = 60$ days.

Claim 4. If we were seeking for a continuous credit policy in an open interval of real numbers, the first order necessary condition would have led us into the following equation:

$$\int_0^{360}(1-\alpha)\frac{dR}{dq}\frac{\partial q}{\partial \tau}dt = \int_0^{360}\frac{dC^u}{du}\frac{\partial u}{\partial \tau}dt + \int_0^{360}\frac{dC^s}{ds}\frac{\partial s}{\partial \tau}dt + \int_0^{360}\frac{dC^q}{dq}\frac{\partial q}{\partial \tau}dt\ . \tag{7}$$

The term in the left may be recognized as cumulated marginal revenue and the term in the right, cumulated marginal cost. Thus, equation (7) is consistent to standard Microeconomics condition (marginal revenue equals marginal cost), however from a cumulative perspective.

Taken together, equations (3), (4), (5’), (6’) and (7) extends Neoclassical theory of the firm from static to dynamic context, considering explicitly the question of finding not only optimal production schedule, but also optimal credit and inventory (finished goods and raw material) policies, consistent to corporate finance setting (value maximization from operating cash flow). Besides, equations (3), (4), (5'), (6') and (7) may be used for planning and control by managers.

6. Operating Cash Flow Determination

Optimal policies (production, inventory and credit) take part in operating cash flow pattern determination. This pattern explains cash needs, therefore the initial investment in current assets or working capital.

6.1. Receipts from Clients

Receipts from clients are total revenue deferred by credit period ($\tau = 60$ days). Denoting receipts from clients as F^+, we have:

$$F^+(t) = \begin{cases} 0, t \in [0,60] \\ R(q(t-60,60)), t \in]60,420] \end{cases}.$$

Since total revenue is $R(t) = pq(t)$, $p = 50$ and $q(t) = 20 + 10\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)$ for $\tau = 60$, yields:

$$F^+(t) = \begin{cases} 0, t \in [0,60] \\ 1000 + 500\cos\left[-\frac{\pi}{2} + \frac{\pi}{15}(t-60)\right], t \in]60,420] \end{cases}.$$

As cosine is a periodic function of period 2π, for each t in [60,420], we may rewrite:

$$F^+(t) = 1000 + 500\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right).$$

6.2. Payment to Suppliers

Recalling equation (4), $\dot{x} = z - \frac{VC^u(u)}{w}$. However, $x(t) = \begin{cases} 0, t = 0 \; or \; t = 360 \\ 4, t \in [0,360] \end{cases}$. Thus, time derivative of x is null, which results $z = \frac{VC^u(u)}{w}$ as raw material unit purchase requirement or $wz = VC^u(u)$ as raw material purchase requirement in monetary units.

Substituting optimal production policy (taken at optimal credit period $\tau = 60$) into variable production cost, we get:

$$
\begin{aligned}
VC^u\big(u(t)\big) = {} & 25\cos^2\left(-\frac{\pi}{2}+\frac{\pi}{15}t\right)+\frac{25\cosh^2 2t}{\sinh^2 720} \\
& - 50\cos\left(-\frac{\pi}{2}+\frac{\pi}{15}t\right)\frac{\cosh 2t}{\sinh 720}+180\cos\left(-\frac{\pi}{2}+\frac{\pi}{15}t\right) \\
& - \frac{180\cosh 2t}{\sinh 720}+260\,.
\end{aligned}
$$

In a more compact form, we may rewrite the above expression as the following:

$$VC^u\big(u(t)\big) = \left[\left(18-\frac{5\cosh 2t}{\sinh 720}\right)+5\cos\left(-\frac{\pi}{2}+\frac{\pi}{15}t\right)\right]^2-64\,.$$

Let F_S^- denote payment to suppliers. Then, supposing suppliers are paid on time, we have:

$$F_S^-(t) = \begin{cases} 0, t = 0 \\ VC^u\big(u(t')\big)+20, t = t' \mathit{for\ some}\ t' \in\,]0,1] \\ VC^u\big(u(t)\big), t \in\,]0,360[\ \mathit{except}\ t' \\ VC^u\big(u(360)\big)-20, t = 360 \end{cases}.$$

The jump (20 = wx, w = 5, x = 4) comes from the discontinuity of raw material inventory at time dates 0 and 360, i.e., initial and terminal conditions.

For each t in [60,360], payment to suppliers has the following upper limit:

$$F_S^-(t) \le \left[18+5\cos\left(-\frac{\pi}{2}+\frac{\pi}{15}t\right)\right]^2-64\,.$$

6.3. Payment due to Rents and Salaries

By definition, for each t in]0,360[, function ψ is the next expression:

$$\psi(t) \stackrel{\text{def}}{=} C^y\big(y(t)\big)+C^x\big(x(t)\big)+C^s\big(s(t)\big)+C^q\big(q(t)\big)+ [F^u-(D^y+D^x+D^s+D^q)]\,.$$

Function ψ is an auxiliary one which will help to define payment due to rents and salaries.

Substituting trajectories into cost functions, we obtain the fourth components of function ψ:

$$C^y\big(y(t)\big) = \begin{cases} 0, t = 0 \\ \frac{25\sinh^2 2t}{\sinh^2 720}+29, t \in\,]0,360] \end{cases}$$

$$C^x(x(t)) = \begin{cases} 0, t = 0 \\ 49, t \in]0,360[\\ 65, t = 360 \end{cases}$$

$$C^s(s(t)) = 111.404, t \in]0,360], s(t,60) \equiv 60000$$

$$C^q(q(t)) = 65 + 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right), t \in]0,360] \ .$$

The remaining components of function ψ are:

$F^u = 20$ (Sum of all salaries and per hour workers at production department)

$D^y = 10$ (Depreciation expense at finished goods inventory facilities)

$D^x = 15$ (Depreciation expense at raw material inventory facilities)

$D^s = 10$ (Depreciation expense at finance department)

$D^q = 15$ (Depreciation expense at sales department).

Then, $F^u - (D^y + D^x + D^s + D^q) = -30$. Depreciation expenses must be excluded because they don't affect cash, although they diminish profit.

Adding all the elements of function ψ, we get

$$\Psi(t) = \begin{cases} 0, t = 0 \\ \frac{25\sinh^2 2t}{\sinh^2 720} + 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 224.404, t \in]0,360[\\ 265.404, t = 360 \end{cases} .$$

Supposing that rents and salaries are paid 30 days after its occurrence, we arrive at payment due to rental and salaries as:

$$F_{RS}^-(t) = \begin{cases} 0, t \in [0,30] \\ \frac{25\sinh^2 2(t-30)}{\sinh^2 720} + 20\cos\left[-\frac{\pi}{2} + \frac{\pi}{15}(t-30)\right] + 224.404, t \in]30,390[\\ 265.404, t = 390 \end{cases}$$

Recalling cosine is 2π periodic function, payment due to rents and salaries has the following upper limit:

$$F_{RS}^-(t) < \begin{cases} 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 225, t \in]60,360] \\ 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 266, t \in]360,390] \end{cases} .$$

6.4. Tax Payment

As done in previous item, we will build an auxiliary function, named tax function, which will help to determine tax payment. As taxes are the sum of revenue tax (indirect tax) with income tax (direct tax), we have the following development:

$Tax(t) = RTax + ITax$

$Tax(t) = \alpha R + \beta Income$, since $RTax = \alpha R$ and $ITax = \beta Income$

However, $Income = (1-\alpha)R - [(C^u + C^y + C^x + C^s + C^q) - \frac{d}{dt}FGI]$ since cost of goods sold is $CGS = C^u - \frac{d}{dt}FGI$, where the term after subtraction sign means the change in finished goods inventory (instantly variation in continuous time). The expression of *GCS* comes from the following accounting identity: $\frac{d}{dt}FGI = C^u - CGS$ (in monetary units, change in finished goods inventory increases with production cost and decreases with cost of goods sold). Thus, after rearranging and grouping terms, we get:

$$Tax(t) = [1-(1-\alpha)(1-\beta)]R\big(q(t)\big) - \beta\big[C^u\big(u(t)\big) + C^y\big(y(t)\big) + C^x\big(x(t)\big) + C^s\big(s(t)\big) + C^q\big(q(t)\big)\big] + \beta\frac{d}{dt}FGI(t)\ .$$

Since $\alpha = 20\%$, $\beta = 40\%$, and $R\big(q(t)\big) = 1000 + 500\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right), \forall t \in]0,360]$, the revenue component of Tax(t) is:

$$[1-(1-\alpha)(1-\beta)]R\big(q(t)\big) = 520 + 260\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right).$$

Define total cost as $(t) = C^u\big(u(t)\big) + C^y\big(y(t)\big) + C^x\big(x(t)\big) + C^s\big(s(t)\big) + C^q\big(q(t)\big)$. Recalling cost components are:

$$C^u(u(t)) = \left[\left(18 - \frac{5\cosh 2t}{\sinh 720}\right) + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 - 14, \forall t \in]0,360]$$

$$C^y\big(y(t)\big) = \begin{cases} 0, t = 0 \\ \frac{25\sinh^2 2t}{\sinh^2 720} + 29, t \in]0,360] \end{cases}$$

$$C^x(x(t)) = \begin{cases} 0, t = 0 \\ 49, t \in]0,360[\\ 65, t = 360 \end{cases}$$

$$C^s\big(s(t)\big) = 111.404, t \in]0,360], s(t,60) \equiv 60000\ .$$

$$C^q\big(q(t)\big) = 65 + 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right), t \in]0,360]\ .$$

Then, cost function results:

$$C(t) = \begin{cases} \left[\left(18 - \frac{5\cos h2t}{\sin h720}\right) + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 + \end{cases}$$

$$0, t = 0$$

$$\frac{25\sin h^2 2t}{\sin h^2 720} + 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 240.404, t \in]0,360[\,.$$

$$450.404, t = 360$$

As β = 40%, the cost component of tax before adjusting for change in finished goods inventory is:

$$\beta C(t) = \begin{cases} 0{,}4\left[\left(18 - \frac{5\cos h2t}{\sin h720}\right) + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 + \end{cases}$$

$$0, t = 0$$

$$\frac{10\sin h^2 2t}{\sin h^2 720} + 8\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 96.1616, t \in]0,360[$$

$$180.1616, t = 360$$

Let Tax*(*t*) be tax function before adjusting for change in finished goods inventory. Then, subtracting cost component from revenue component, yields:

$$Tax^*(t) = \begin{cases} -0{,}4\left[\left(18 - \frac{5\cos h2t}{\sin h720}\right) + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 - \end{cases}$$

$$0, t = 0$$

$$\frac{10\sin h^2 2t}{\sin h^2 720} + 252\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 423.8384, t \in]0,360[$$

$$339.8384, t = 360$$

Taking into account the adjustment for finished goods inventory, we can right:

$$Tax(t) = Tax^*(t) + \beta\frac{d}{dt}FGI(t)\,.$$

However, when measured in terms of FIFO (First In, First Out) criterion, finished goods inventory in monetary units results in average production cost times finished goods inventory:

$$FGI(t) = AC^u\big(u(t)\big)y(t), AC^u(u) = \frac{C^u(u)}{u}\,.$$

For time derivative of *FGI*(*t*), we have

$$\frac{d}{dt}AC^u\big(u(t)\big)y(t) = [MgC^u(u) - AC^u(u)]\frac{\dot{u}}{u}y + AC^u(u)\dot{y} .$$

Recalling $C^u(u) = \frac{1}{4}u^2 + 8u + 50$, then $MgC^u(u) - AC^u(u) = \frac{1}{4}u - \frac{50}{u}$, thus we have:

$$\beta\frac{d}{dt}FGI(t) = 0{,}4 * \left[\left(\frac{1}{4}u - \frac{50}{u}\right)\frac{\dot{u}}{u}y + AC^u(u)\dot{y}\right] < 0{,}1\dot{u}y .$$

The above inequality follows because

$\frac{1}{4}u - \frac{50}{u} < \frac{1}{4}u, \forall u > 0, AC^u(u) = \frac{1}{4}u + 8 + \frac{50}{u} > 0, \forall u > 0$ and

$$\dot{y} = -\frac{10\cosh 2t}{\sinh 720} < 0, \forall t > 0 .$$

Recalling *u* and *y*, we get

$$\dot{u}(t) = -\left[\frac{20\sinh 2t}{\sinh 720} + \frac{2\pi}{3}\sin\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right] \Longrightarrow \dot{u}(t) \leq \frac{2\pi}{3}, \forall t \in [0,360]$$

$$y(t) = 5\left(1 - \frac{\sinh 2t}{\sinh 720}\right) \Longrightarrow y(t) \leq 5, \forall t \in [0,360] .$$

Therefore

$$\beta\frac{d}{dt}FGI(t) < 0{,}1\dot{u}y \leq \frac{\pi}{3} < 2 .$$

Finally, supposing taxes are paid after 30 days and recalling cosine is 2π periodic function, we conclude tax payment has the following upper limit:

$$F_T^-(t) < -0{,}4\left[13 + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 + 252\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 426, t \in]30,390] .$$

6.5. Operating Cash Flow Pattern

Operating Cash Flow is defined as follows:

$$OCF(t) \overset{\text{def}}{=} \begin{cases} 0, t = 0 \\ -F_S^-(t), t \in]0,30] \\ -[F_S^-(t) + F_{RS}^-(t) + F_T^-(t)], t \in]30,60] \\ F^+(t) - [F_S^-(t) + F_{RS}^-(t) + F_T^-(t)], t \in]60,360] \\ F^+(t) - [F_{RS}^-(t) + F_T^-(t)], t \in]360,390] \\ F^+(t), t \in]390,420] \end{cases}$$

According to the above definition, we state:

(i) $OCF(t) < 0, \forall t \in]0,60]$

(ii) $OCF(t) < 0, \forall t \in]60,420]$.

Proof:

(i) Trivial, since $F^+(t) = 0, F_S^-(t) > 0, F_{RS}^-(t), F_T^-(t) \geq 0, \forall t \in]0,60]$.

(ii) Trivial for $OCF(t) = F^+(t) > 0, \forall t \in]390,420]$. So, we must analyze the following:

Case 1:]60,360]

We have

$$F^+(t) = 1000 + 500\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)$$

$$F_S^-(t) \leq \left[18 + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 - 64$$

$$F_{RS}^-(t) < 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 225$$

$$F_T^-(t) < -0{,}4\left[13 + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 +$$

$$252\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 426 .$$

Then

$$OCF(t) > -15\cos^2\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 100\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 156{,}6 \geq$$

$$-15 - 100 + 156{,}6 = 41{,}6 > 0 .$$

Case 2:]360,390]

We have

$$F^+(t) = 1000 + 500\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)$$

$$F_{RS}^-(t) < 20\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 266$$

$$F_T^-(t) < -0{,}4\left[13 + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 +$$

$$252\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + 426$$

$$OCF(t) > 0{,}4\left[13 + 5\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\right]^2 + 228\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) +$$

$308 > -228 + 308 = 80 > 0$.

Therefore, taking together cases 1, 2 and trivial (ii), we conclude $OCF(t) > 0, \forall t \in]60,420]$.

7. Working Capital Determination

At the very beginning, initial current assets equals initial cash and initial current liabilities is null, because none operation has already taken place (firm is still planning). So, initial working capital corresponds to initial cash M_0. Then, the task is to find initial cash bearing in mind two elements: (1) operating cash flow pattern (item 6.5) and (2) liquidity-return trade off (risk-return profile).

Recognizing that change in money demand (cash holdings) equals operating cash flow, money demand has the following structure:

$$M(t) = M_0 + \int_0^t OCF(\xi)\, d\xi, \forall t \in]0{,}420] .$$

Since operating cash flow is strictly negative in]0,60] and becomes strictly positive in]60,420], cumulated change in cash (definite integral) reaches a global minimum at $t = \tau^* = 60$. Observe that $\tau^* = 60$ days correspond to optimal credit period and, in this paper, because suppliers credit period is zero, optimal credit period can be understood as operating cycle as well as financing cycle.

In order to avoid bankruptcy, firm must find M_0 such that $M(t) > 0$ for each t in]0,420]. To achieve this goal, M_0 must be set as follows:

$$M_0 = M_g - \int_0^{60} OCF(t)\, dt, \text{ for some } M_g > -\int_0^{60} OCF(t)\, dt .$$

Recalling Keynes, liquidity preference refers to a situation in which money demand is insensitive (doesn't change with) to current (or expected) interest rate within a certain interest rate range. Putting it into a formal way, we state: $M^d = M_g, \forall r \in [0, r_p]$, for some $r_p > 0$.

Hence, supposing firm owner (or manager) has a conservative risk-return profile such that r_p is larger than any current (or expected) interest rate $r = r(t)$ for all t in]0,420], firm owner (or manager) is found in liquidity preference position. Supposing further that capital restriction is not an active constrain in our model, firm owner (manager) is free to choose any targeted M_g which he feels safety enough in light of his risk-liquidity (risk-return) profile. For example, targeted M_g could be chosen as fifty percent of (expected) maximum cash needs (the negative of cumulated *OCF* global minimum), i.e., $M_g = -0.5 \int_0^{60} OCF(t)\, dt$.

Therefore, recalling Keynes again, firm money demand (cash holdings) splits into two components: (1) initial cash (M_0) and (2) cumulated *OCF*. The first is chosen according to precautionary motive (as a percentage of the negative of cumulated *OCF* global minimum). The second is explained by transaction motive, i.e., the quest for optimal value operating policies (credit period, production, finished goods and raw material inventories), given a forecasted demand.

We finish this section determining working capital numerically. In light of the operating cash definition, cumulated cash flow global minimum is given by

$$\int_0^{60} OCF(t)\,dt = \int_0^{30} -F_S^-(t)\,dt + \int_{30}^{60} -[F_S^-(t) + F_{RS}^-(t) + F_T^-(t)]\,dt\ .$$

Thus, expected cash needs are:

$$-\int_0^{60} OCF(t)\,dt = \int_0^{60} F_S^-(t)\,dt + \int_{30}^{60} F_{RS}^-(t)\,dt + \int_{30}^{60} F_T^-(t)\,dt\ .$$

As cash flow payments, we have:

i) Payment to suppliers in]0,60]:

$$\begin{aligned} F_S^-(t) = {} & 25\cos^2\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) + \frac{25\cos h^2 2t}{\sin h^2 720} \\ & - 50\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right)\frac{\cos h2t}{\sin h720} + 180\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right) \\ & - \frac{180\cos h2t}{\sin h720} + 260\ . \end{aligned}$$

ii) Payment due to rents and salaries in]30,60]:

$$F_{RS}^-(t) = \frac{25\sin h^2 2(t-30)}{\sin h^2 720} + 20\cos\left[-\frac{\pi}{2} + \frac{\pi}{15}(t-30)\right] + 224.404$$

iii) Tax payment in]30,60]:

$$\begin{aligned} F_T^-(t) = {} & -10\cos^2\left[-\frac{\pi}{2} + \frac{\pi}{15}(t-30)\right] - \frac{10\cos h^2 4(t-30)}{\sin h^2 720} \\ & + 20\cos\left[-\frac{\pi}{2} + \frac{\pi}{15}(t-30)\right]\frac{\cos h2(t-30)}{\sin h720} \\ & + 180\cos\left[-\frac{\pi}{2} + \frac{\pi}{15}(t-30)\right] + \frac{72\cos h2(t-30)}{\sin h720} \\ & + 294{,}2384 + 0{,}4\frac{d}{dt}\left[AC^u\big(u(t-30)\big)y(t-30)\right]. \end{aligned}$$

Where in the above expression holds

$$AC^u(u) = \frac{1}{4}u + 8 + \frac{50}{u}, u > 0$$

$$u(t) = \left(20 - \frac{10\cos h2t}{\sin h720}\right) + 10\cos\left(-\frac{\pi}{2} + \frac{\pi}{15}t\right), \forall t \in\ ]0,360]$$

$y(t) = 5\left(1 - \frac{\sinh 2t}{\sinh 720}\right), \forall t \in]0{,}360], y_0 = 0$.

So, the definite integrals result:

$\int_0^{60} F_S^-(t)\, dt = 16350$ $\qquad \int_{30}^{60} F_{RS}^-(t)\, dt = 6732.12$

$\int_{30}^{60} F_T^-(t)\, dt = 8858.152$.

Therefore, expected cash needs are:

$-\int_0^{60} OCF(t)\, dt = 31940.272 = 32000$ (2 significant digits).

Recalling targeted cash holdings and initial cash expressions,

$M_g = -0.5 \int_0^{60} OCF(t)\, dt \quad M_0 = M_g - \int_0^{60} OCF(t)\, dt$.

Initial cash (initial investment in current assets or working capital) is calculated as:

$M_0 = -1.5 \int_0^{60} OCF(t)\, dt = 48000$ (2 significant digits).

In view of optimal operating policies and risk-liquidity consideration, the model concludes that initial investment in current assets accounts for \$48000.

8. Conclusions

This paper addressed the issue of integrating the elements of working capital determination. More specifically, we aimed to answer what would be the amount to invest in initial current assets. In this regard, hanging on a deterministic demand varying linearly with credit period and quadratic cost functions, we found an expression that reveals not only the amount to invest in initial current assets, but the intrinsic structure that determines that amount.

The revealed structure shows that initial investment in current assets (initial cash) depends explicitly on both operating characteristics and risk-liquidity concern. Operating characteristics refers to sales demand and choices firm must make relating to credit, production, finished goods and raw material inventories. They are the factors that determine operating cash flow pattern, hence explains part of initial investment in current assets. Risk-liquidity concern relates to liquidity position to ensure firm will have a minimum cash to pay its obligations timely; therefore, completes the amount of initial investment in current assets.

Operating characteristics adjust dynamically. This is shown in equations (3), (4), (5’), (6’) and (7) which extend neoclassical theory of the firm from static to dynamic context. These equations consider explicitly the question of finding not only optimal production schedule, but also optimal credit and finished goods and raw material inventories, in light of value maximization (from operating cash flow). Besides, equations (3), (4), (5’), (6’) and (7) may be used for planning and control.

The model developed also shows that setting inventory at its minimal cost is not always the case. So managers need not rely on business practices which advocates minimum inventory. Instead, they are able to check if an inventory policy leads to maximum value using equations (5’) and (6’). In fact, minimal cost inventory is optimal only if value function is constant with respect to inventory control variable, a very restrictive condition. In this regard, the model developed in this paper is a counterexample to minimum cost inventory practices. Yet, another counterexample is shown in accounts receivable. Minimum cost of accounts receivable is achieved when accounts receivable is \$1000. However, optimal credit period (of 60 days) requires accounts receivable be at \$60000 (sixty times more than that minimum).

We did not take into account explicitly the question of uncertainty. Then, a more accurate result when dealing with risk-liquidity concern emerges considering uncertainty explicit. One possible way to do that is to introduce stochastic processes, for example, in demand. Besides, introduce formally risk-liquidity optimization problem. Another possible shortcoming of this paper may be related to simultaneity of events. Specifically we only consider credit period as a source of operating cycle. However, raw material purchase, production, storage and sales, all have a specific time span. Thus, our model represents cash inflows and cash outflows as daily occurrences with no serious time gaps or desynchronization. These issues are proposed for future research.

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