

# WILLINGNESS TO PAY IN THE THEORY OF A CONSUMER

Alia Asha DANNENBERG\*, Matti ESTOLA\*\*

***Abstract.** Current neoclassical theory of a consumer has some shortcomings that hinder the mathematical modelling of real consumer behaviour. Being based on static optimisation, the neoclassical theory is unable to cover dynamic effects, and its results are valid only in the optimum point. It is also confusing with respect to the concepts of utility, money, willingness to pay, and demand. Here we introduce a new theory for consumer behaviour expressed with measurable quantities that enables both empirical testing and forecasting. A consumer's adjustment is modelled outside possible equilibrium points as well to explain a consumer's movement from one equilibrium to another. Our theory allows time-dependent variables that are necessary for developing a dynamic economic theory, and the neoclassical theory is shown to be a special case of our model. PACS: 89.65. GhEconophysics, 89.65.-s Social and economic systems, 89.75.-k Complex systems.*

## 1. Background

In generating a theory of consumer behaviour, the greatest problems are involved with the “free will” of humans [1, 2]. The neoclassical framework is based on the assumption that humans behave in a rational (or boundedly rational) way [3, 4]. A rational utility-seeking consumer is assumed to maximise his/her utility with respect to some constraints. A utility-seeking consumer adjusts his/her consumption towards the direction where the *efficiency of spending*<sup>1</sup> is the highest, i.e., where the ratio of marginal utility and price of every good is equal [2], [5]: p. 170.

---

\* Lukema Oy, Joensuu, Finland, e-mail: [alia.dannenberg@lukema.eu](mailto:alia.dannenberg@lukema.eu)

\*\* University of Eastern Finland, Faculty of Social Sciences, P.O. Box 111, FIN-80101 Joensuu, Finland, e-mail: [matti.estola@uef.fi](mailto:matti.estola@uef.fi)

<sup>1</sup> Through the article, with *efficiency of spending* we mean "the efficiency of gaining utility from spending money on a certain good".

Within the neoclassical framework, the adjustment of a consumer toward his/her optimum has been assumed implicitly (see e.g., Ref. [5]: p. 174), but the adjustment process has not been modelled formally. This shortcoming can be illustrated by the following analogy: suppose a concave cup and a marble on its edge. What happens when the marble is let loose? In physics, we can write the equations of motion of the marble and calculate its trajectory as a solution of these equations. In economics, the best the neoclassical theory can do is to tell that after some time the marble will be immobile in the bottom of the cup. Thus the neoclassical framework cannot explain which trajectory of a consumer led him/her into his/her optimal position, how long did the adjustment take, and whether there occurred overshooting or not.

The testing of the neoclassical theory of a consumer is also difficult because there are no means to directly measure a consumer's marginal utilities of goods or his/her level of utility. Our solution to this is that instead of marginal utility, we apply the concept of a consumer's *marginal willingness to pay for a good* that can be measured [6, 7, 8, 9] e.g. by making a consumer survey. Moreover, we show that utility function has an impact on consumer behaviour only via the marginal willingness to pay. Thus the actual measurement of utility is not needed in modelling consumer behaviour.

Although the concept of willingness to pay is widely used in literature [5, 6, 7, 8, 9, 10, 11], and it is linked to demand [5, 8, 10], to the shadow prices of possible constraints [9, 11], and to compensating variation [12]: Chapter 3, [13]: Chapter 3, the concept is lacking a consistent conceptual definition. Hence, we define the concept "marginal willingness to pay"<sup>2</sup> in a new way. This way we get the results of the theory in a testable form, and it enables the calculation of consumer surplus in monetary units. Marginal willingness to pay is required for consistent definitions and relations between utility, marginal utility, and willingness to pay.

We model mathematically how a rational consumer adjusts his/her behaviour outside his/her optimum, and we give fundamentals on which

---

<sup>2</sup> The concept "marginal willingness to pay" has been applied in studies of environmental and public sector economics, see e.g., Refs. [14, 15, 16]. However, it has only been used in evaluating the value of public services that are lacking market price [14], and in estimating the price premium a consumer is willing to pay for e.g. an eco-certified good [15, 16]. In Ref. [16], marginal willingness to pay is defined as "the income reduction needed to ensure constant utility when an attribute is added in the marginal good". This results from thinking willingness to pay as compensating variation, which seems confusing in the light of our present analysis.

one can build trajectory calculations outside optimum<sup>3</sup>. Our aim is to give a theory that allows modelling and forecasting the behaviour of a group of heterogeneous consumers by statistical means. This is similar to statistical physics where e.g. the macroscopic properties of gas (temperature, pressure, etc.) depend on the interactions of (possibly heterogeneous) gas molecules. We can think that a group of heterogeneous consumers behaves like a group of heterogeneous molecules, yielding statistical behaviour that arises from the properties of the group and the interactions between the agents [17, 18]. However, the statistical behavior of a group of consumers will be exactly modelled in another study.

The needs of humans are highly similar, and thus there are certain statistical regularities in the group level human behaviour (like the need of food and sleep). An essential part in our modelling is to allow a consumer specific willingness to pay for every good, and thus not to make the restricting assumption that all consumers have identical preferences. This restriction is common in representative agent models [19], and it has been criticised e.g. in Refs. [20, 21]<sup>4</sup>. By allowing heterogeneous preferences we do not restrict a consumer's behaviour at all. The only “restriction” in our modelling is that a consumer is consistent with his/her own preferences. A group of heterogeneous consumers creates a distribution of marginal willingness to pay for every good, and the group level behaviour can then be modelled by statistical means.

Through the article we stress the principle of consistency of measurement units of economic quantities [23], and thus we express the units explicitly even though they may appear implicitly clear.

## 2. The neoclassical theory of a consumer

Let a consumer's optimisation problem for an  $n$ -product system be: Maximise utility function  $U(\mathbf{X})$  (*utility/time*) with respect to budget

---

<sup>3</sup> In real economic studies, trajectory calculus is important e.g. in showing how opening up the former Eastern bloc affects the development of these countries. During the adjustment toward the market system, the neoclassical framework is of no use since it will tell nothing about the process. It can only claim that e.g. after 40 years these economies will be in the neoclassical market equilibrium. The policy makers would have to wait this time before they could start using the tools developed within the neoclassical framework.

<sup>4</sup> The representative agent model has been applied in DSGE models (dynamic stochastic general equilibrium) by allowing the parameters of the “agent” to change with the applied economic policy. This makes the representative agent more “flexible” [22].

inequality  $M \geq \sum_{i=1}^n P_i X_i$ , where  $\mathbf{X} = (X_1, \dots, X_n)$  is the vector of consumption flows of goods of the consumer. The unit of  $X_i$  is *piece<sub>i</sub>/time*, that of price  $P_i$  is  $\text{€}/\text{piece}_i$ , and  $M$  ( $\text{€}/\text{time}$ ) is the income of the consumer<sup>5</sup>. The problem can be expressed as the Lagrangian

$$\max_{\mathbf{X}} L = U(\mathbf{X}) + \lambda[M - \sum_{i=1}^n P_i X_i], \quad (1)$$

where  $\lambda$  is the Lagrangian multiplier with unit *utility/€* for dimensional consistency [23]. The Kuhn-Tucker first order conditions for optimum are (see e.g., Ref. [12]: p. 53-55)

$$\frac{\partial L}{\partial X_i} = \frac{\partial U}{\partial X_i} - \lambda P_i \leq 0, \quad X_i \geq 0, \quad \frac{\partial L}{\partial X_i} X_i = 0 \quad \forall i, \quad (2)$$

$$M - \sum_{i=1}^n P_i X_i \geq 0, \quad \lambda \geq 0, \quad \lambda[M - \sum_{i=1}^n P_i X_i] = 0. \quad (3)$$

Assuming  $X_i > 0, \forall i$ , Eqs. (2) yield

$$\frac{1}{P_1} \frac{\partial U}{\partial X_1} = \dots = \frac{1}{P_n} \frac{\partial U}{\partial X_n} = \lambda, \quad (4)$$

which condition holds in every optimum of the consumer if  $X_i > 0, \forall i$ . Eq. (3) shows that in a border optimum holds  $M = \sum_{i=1}^n P_i X_i$ , and in an inner point optimum holds  $\lambda = 0$  [12]: p. 958-964. In a border optimum,  $\lambda$  measures the marginal utility of income (or the shadow price of the budget constraint) [12]: p. 54. Quantity  $(1/P_i) \partial U / \partial X_i$  with unit<sup>6</sup> *utility/€* measures the consumer's efficiency in receiving utility from the consumption of good  $i$  per one euro. In the optimum, every good has an equal efficiency in giving utility.

The dual expenditure minimisation problem of a consumer can be expressed as (Ref. [12]: p. 57-63)

$$\min_{\mathbf{X}} W = \sum_{i=1}^n P_i X_i + \mu[u_0 - U(\mathbf{X})], \quad (5)$$

where  $U(\mathbf{X}) \geq u_0$  and  $\mu$  is the Lagrangian multiplier with unit  $\text{€}/\text{utility}$  for dimensional consistency. The Kuhn-Tucker first order conditions for optimum are

---

<sup>5</sup> The unit of time can be e.g. one day or one week. *Piece* is an arbitrary unit of volume of a good, and it can refer to a suitable SI unit.

<sup>6</sup> The unit of a partial derivative can be derived according to its definition, i.e.  $\partial U / \partial X_i = \lim_{\Delta X_i \rightarrow 0} \Delta U / \Delta X_i$ , where the unit of  $\Delta U$  is *utility/time* and that of  $\Delta X_i$  is *piece<sub>i</sub>/time*, and taking the limit does not affect the measurement unit.

$$\frac{\partial W}{\partial X_i} = P_i - \mu \frac{\partial U}{\partial X_i} \geq 0, \quad X_i \geq 0, \quad \frac{\partial W}{\partial X_i} X_i = 0 \quad \forall i, \quad (6)$$

$$U(\mathbf{X}) - u_0 \geq 0, \quad \mu \geq 0, \quad \mu[u_0 - U(\mathbf{X})] = 0. \quad (7)$$

In an inner point optimum  $\mu = 0$ , and in a border optimum  $U(\mathbf{X}) = u_0$ .

Assuming binding budget equation in the utility maximisation problem (1) and  $X_i > 0, \forall i$ , we can solve the Marshallian demand functions  $X_{iM} = f_i(\mathbf{P}, M), \forall i$ , and the value of the Lagrangian multiplier  $\lambda^* = f(\mathbf{P}, M)$ . A border optimum in the expenditure minimisation problem (5) yields the Hicksian demand functions  $X_{iH} = g_i(\mathbf{P}, u_0), \forall i$ , and the value of the Lagrangian multiplier  $\mu^* = g(\mathbf{P}, u_0)$ . These demand equations are valid only in a border optimum, and in the optimum holds  $X_{iM} = X_{iH}, \forall i, \lambda^* = 1/\mu^*$ , and  $\partial U / \partial M = \lambda^*$ .

**Example 1.** Let the Cobb-Douglas form for utility with two goods  $U = A(X_1/X_{10})^\alpha(X_2/X_{20})^{1-\alpha}$ , where  $0 < \alpha < 1$  is a pure number, constant  $A > 0$  has unit *utility/time*, and  $X_{i0}, i = 1, 2$  are constant initial consumptions of the two goods with units *piece<sub>i</sub>/time*, respectively. This dimensionally correct form [23] for utility can be presented as  $U = BX_1^\alpha X_2^{1-\alpha}$ , where constant  $B = AX_{10}^{-\alpha} X_{20}^{\alpha-1} > 0$  has unit *utility*  $\times$  *piece<sub>1</sub><sup>- $\alpha$</sup>*   $\times$  *piece<sub>2</sub> <sup>$\alpha-1$</sup>* . Even though in dimensional analysis measurement units with non-integer powers (e.g.  $kg^{0.7}$ ) are not allowed [23]: pp. 46-47, we still operate with  $B$  as if it were a well-defined quantity. This assumption simplifies the calculations while still giving correct units for all relevant quantities:  $X_1, X_2, U, v, e, \lambda, \mu$ .

Assuming binding budget equation  $M = P_1 X_1 + P_2 X_2$ , the optimal border solution is:

$$X_{1M} = \frac{\alpha M}{P_1}, \quad X_{2M} = \frac{(1-\alpha)M}{P_2}, \quad \lambda^* = B \left(\frac{\alpha}{P_1}\right)^\alpha \left(\frac{1-\alpha}{P_2}\right)^{1-\alpha},$$

$$v(P, M) = B \left(\frac{\alpha}{P_1}\right)^\alpha \left(\frac{1-\alpha}{P_2}\right)^{1-\alpha} M,$$

$$X_{1H} = \frac{u_0}{B} \left(\frac{\alpha P_2}{(1-\alpha)P_1}\right)^{1-\alpha}, \quad X_{2H} = \frac{u_0}{B} \left(\frac{(1-\alpha)P_1}{\alpha P_2}\right)^\alpha, \quad \mu^* = \frac{1}{B} \left(\frac{P_1}{\alpha}\right)^\alpha \left(\frac{P_2}{1-\alpha}\right)^{1-\alpha}, \quad (8)$$

$$e(P, u_0) = \frac{u_0}{B} P_1^\alpha P_2^{1-\alpha} (\alpha^{-\alpha} (1-\alpha)^{\alpha-1}).$$

The units of  $X_{1M}$  and  $X_{1H}$  e.g. are *piece<sub>1</sub>/time* and  $\lambda^*$  has unit *utility/€* and that of  $\mu^*$  is *€/utility*.  $v(P, M) = U(X_M)$  with unit *utility/time* is the indirect utility function, and  $e(P, u_0) = \sum P_i X_{iH}$  with unit *€/time* the expenditure function [12]: pp. 56-59. These results hold, however, only in a border optimum.

Now, restricting the analysis to a border optimum is not reasonable because a complete theory should also explain how a consumer finds his/her optimum. For every point in the consumption space, one can measure the efficiency of gaining utility from spending money on good  $i$ , regardless of how far the consumer is from his/her optimum. The efficiency of spending on good  $i$  is quantified by

$$\xi_i \equiv \frac{1}{P_i} \frac{\partial U}{\partial X_i} \quad (9)$$

with unit *utility*/€. With a binding budget and the consumer being in the optimum, all  $\xi_i$ 's are equal with  $\lambda$ . Outside optimum, vector  $\xi$  can be used as the "compass needle" that points towards the consumer's optimum. The adjustment rules are: increase the consumption of good  $i$ , or decrease that of good  $j$ , if

$$\xi_i = \frac{1}{P_i} \frac{\partial U}{\partial X_i} > \frac{1}{P_j} \frac{\partial U}{\partial X_j} = \xi_j. \quad (10)$$

The problem here is that even if a consumer is aware of his/her efficiency values for all goods, due to their measurement unit, it would be impossible for an observer to quantify them. Thus in the neoclassical theory, a consumer's adjustment outside optimum has a measurement problem that prohibits its direct empirical testing<sup>7</sup>.

For this reason, we introduce a new idea how to model a consumer's adjustment outside his/her optimum. Suppose a consumer is not in his/her optimum, and he/she wants to improve his/her situation; this occurs, e.g. when the consumer faces a price or an income change. Eq. (2) shows that close enough to the border optimum (where  $\lambda^* > 0$ ), a utility-seeking consumer increases  $X_i$  if  $\partial U / \partial X_i > \lambda P_i \Leftrightarrow (\partial U / \partial X_i) / \lambda > P_i$ , and vice versa. Similarly, Eq. (6) shows that close enough to the border optimum (where  $\mu^* > 0$ ), an expenditure-minimising consumer increases  $X_i$  if  $\mu(\partial U / \partial X_i) > P_i$ , and vice versa.

We can interpret  $(\partial U / \partial X_i) / \lambda$  with unit €/piece <sub>$i$</sub>  as this consumer's *marginal willingness to pay* for good  $i$  at his/her current consumption  $\mathbf{X}$ .

---

<sup>7</sup> In the neoclassical framework, it is possible to test consumer behavior by testing goods in pairs by questioning: "Would you prefer good  $i$  with price  $P_i$  to good  $j$  with price  $P_j$ ? This procedure is known as *revealing preferences*." It has been shown, e.g., in Ref. [12]: pp. 10-14, 28-35, that with certain assumptions, a rational preference ordering with known prices and income is complete and fulfills the weak axiom of revealed preferences. However, forcing a consumer to find his/her willingness to pay for a good by the pairwise comparison of it to all other goods is very tedious as compared with our solution.

The reason for this is that a utility-seeking consumer increases the consumption of good  $i$  if  $(\partial U / \partial X_i) / \lambda > P_i$ , and in increasing the consumption of good  $i$ , he/she actually pays the price of good  $i$ . Thus, the willingness to increase the consumption of a good with a known price is equivalent to that the consumer is willing to pay at least the price of the good. In a border optimum, the two marginal willingness to pay concepts are equal with the price:  $(\partial U / \partial X_i) / \lambda^* = \mu^* (\partial U / \partial X_i) = P_i$ .

Thus, a utility-seeking (or an expenditure-minimising) consumer applies the following adjustment rule: *compare your marginal willingness to pay with the price of a good, and increase (decrease) the flow of consumption of the good if your marginal willingness to pay exceeds (is below) the price*. All quantities in these comparisons have clear measurement units, and in this form an observer can test the theory by making a survey of consumers' marginal willingness to pay of the goods of interest.

One should notice that the marginal willingness to pay for a good depends on a consumer's current satisfaction of needs. Think of a consumer's marginal willingness to pay for petrol before and after buying a car. Moreover, in our framework a consumer's preferences are no more restricted than in the neoclassical one.

### 3. Marginal willingness to pay and demand

In textbooks of economics, the demand and the marginal willingness to pay functions are considered to be equal. Ref. [24]: p. 298: "The demand curve reflects a consumer's marginal willingness to pay: the maximum amount a consumer will spend for an extra unit". We show, however, that a consumer's marginal willingness to pay and Marshallian demand functions deviate from each other. Subsequently, a consumer's willingness to pay function is denoted by  $H$  and his/her marginal willingness to pay function for good  $i$  by  $\partial H / \partial X_i$ .

Let a consumer's utility function  $U(\text{utility}/\text{time})$  and budget equation with two consumer goods and income  $M$  ( $\text{€}/\text{time}$ ) in a time unit be

$$U = f(X_1, X_2), \quad M = P_1 X_1 + P_2 X_2. \quad (11)$$

Substituting the budget equation in the utility function (we thus assume the budget equation to hold) yields

$$U = f(X_1, X_2) = f\left(X_1, \frac{M - P_1 X_1}{P_2}\right) \equiv F(X_1, M, P_1, P_2). \quad (12)$$

The consumer's marginal willingness to pay for good 1 is then

$$\frac{\partial H}{\partial X_1} = \frac{\frac{\partial f(X_1, X_2)}{\partial X_1}}{\lambda} = \frac{\frac{\partial f(X_1, X_2)}{\partial X_1}}{\frac{1}{P_2} \frac{\partial f(X_1, X_2)}{\partial X_2}}, \quad \lambda = \frac{\partial U}{\partial M} = \frac{1}{P_2} \frac{\partial f(X_1, X_2)}{\partial X_2}, \quad (13)$$

and the following conditions

$$\frac{\partial^2 f}{\partial X_1^2} < 0, \quad \frac{\partial^2 f}{\partial X_2^2} < 0, \quad \frac{\partial^2 f}{\partial X_1 \partial X_2} = \frac{\partial^2 f}{\partial X_2 \partial X_1} > 0, \quad (14)$$

guarantee that the equilibrium point is a maximum. The following equation corresponds to the optimum:

$$P_1 = \frac{\partial H}{\partial X_1} \Leftrightarrow P_1 = P_2 \frac{\frac{\partial f(X_1, X_2)}{\partial X_1}}{\frac{\partial f(X_1, X_2)}{\partial X_2}}. \quad (15)$$

Eq. (15) is the *Marshallian inverse demand function for good 1 of the consumer*. This relation is similar to that of the marginal willingness to pay, but their slopes in coordinate system  $(X_1, \text{€/piece}_1)$  differ. By totally differentiating Eq. (15) and using the utility function in Eq. (12), we get (see Ref. [25]: pp. 126-134)

$$\begin{aligned} & \left( 1 + \frac{\left( \frac{\partial^2 f}{\partial X_2 \partial X_1} \frac{\partial f}{\partial X_2} \frac{\partial^2 f}{\partial X_2^2} \frac{\partial f}{\partial X_1} \right) X_1}{\left( \frac{\partial f}{\partial X_2} \right)^2} \right) dP_1 = \\ & \left( \frac{\left( \frac{\partial^2 f}{\partial X_1^2} \frac{\partial^2 f}{\partial X_2 \partial X_1} \frac{P_1}{P_2} \right) \frac{\partial f}{\partial X_2} - \left( \frac{\partial^2 f}{\partial X_1 \partial X_2} \frac{\partial^2 f}{\partial X_2^2} \frac{P_1}{P_2} \right) \frac{\partial f}{\partial X_1}}{\left( \frac{\partial f}{\partial X_2} \right)^2} P_2 \right) dX_1 \\ & + \left( \frac{\frac{\partial^2 f}{\partial X_2 \partial X_1} \frac{\partial f}{\partial X_2} \frac{\partial^2 f}{\partial X_2^2} \frac{\partial f}{\partial X_1}}{\left( \frac{\partial f}{\partial X_2} \right)^2} \right) dM + \left( \frac{\left( \frac{\partial^2 f}{\partial X_2^2} \frac{\partial f}{\partial X_1} \frac{\partial^2 f}{\partial X_2 \partial X_1} \frac{\partial f}{\partial X_2} \right) \left( \frac{M - P_1 X_1}{P_2} \right) + \frac{\partial f}{\partial X_2}}{\left( \frac{\partial f}{\partial X_2} \right)^2} \right) dP_2. \quad (16) \end{aligned}$$

Eq. (16) can be presented as

$$a_1 dP_1 = a_2 dX_1 + a_3 dM + a_4 dP_2, \quad a_1 > 0, \quad a_2 < 0, \quad a_3 > 0, \quad (17)$$

where  $a_i$ ,  $i = 1, \dots, 4$ , are the coefficients of the differentials;  $a_4$  is of ambiguous sign. Eq. (17) yields

$$\frac{\partial P_1}{\partial X_1} = \frac{a_2}{a_1} < 0, \quad \frac{\partial P_1}{\partial M} = \frac{a_3}{a_1} > 0, \quad \frac{\partial P_1}{\partial P_2} = \frac{a_4}{a_1}, \quad (18)$$

where the sign of the last partial is ambiguous. Because  $P_1$  and  $\partial H / \partial X_1$  both have unit  $\text{€/piece}_1$ , they can be measured on the same coordinate axis. However, the slope  $\partial P_1 / \partial X_1 = a_2 / a_1 < 0$  of the inverse demand

relation (15) in coordinate system  $(X_1, \text{€/piece}_1)$  deviates from that of the marginal willingness to pay:  $\partial^2 H / \partial X_1^2 = a_2 < 0$ . Because  $a_1 > 1$ , the latter curve is steeper. The reason for this is the income effect a change in price has on the marginal willingness to pay. If  $P_1$  decreases, the utility-maximising flow of consumption of good 1 increases. Consequently, a price decrease raises the consumer's real budgeted funds and moves his/her marginal willingness to pay relation away from the origin.

Eqs. (13) and (15) give similar results, and both are useful. The Marshallian demand relation can be estimated from the real world by statistical means with observed prices and flows of consumption, and the marginal willingness to pay relation can be quantified by making a consumer survey.

**Example 2.** Let a consumer's utility function be (as in Example 1)

$$U = BX_1^\alpha X_2^{1-\alpha}, \quad B > 0, \quad 0 < \alpha < 1. \quad (19)$$

Substituting the budget equation  $M = P_1 X_1 + P_2 X_2$  in Eq. (19), we get

$$U = BX_1^\alpha \left( \frac{M - P_1 X_1}{P_2} \right)^{1-\alpha}. \quad (20)$$

The necessary condition for the consumer's optimum is

$$\frac{dU}{dX_1} = 0 \Leftrightarrow \alpha BX_1^{\alpha-1} \left( \frac{M - P_1 X_1}{P_2} \right)^{1-\alpha} - (1 - \alpha) BX_1^\alpha \left( \frac{M - P_1 X_1}{P_2} \right)^{-\alpha} \frac{P_1}{P_2} = 0. \quad (21)$$

From Eq. (21) we get the Marshallian demand and inverse demand functions for good 1 as

$$X_{1M} = \frac{\alpha M}{P_1} \Leftrightarrow P_1 = \frac{\alpha M}{X_{1M}}. \quad (22)$$

If we multiply the first order condition in Eq. (21) by factor

$$\frac{X_1^{-\alpha} \left( \frac{M - P_1 X_1}{P_2} \right)^\alpha P_2}{(1-\alpha)B} > 0 \quad \text{we get} \quad \frac{\alpha}{1-\alpha} \left( \frac{M}{X_1} - P_1 \right) - P_1 = 0, \quad (23)$$

where  $\partial H / \partial X_1 = \alpha / (1 - \alpha) (M / X_1 - P_1)$  is the consumer's marginal willingness to pay function for good 1. *Notice that in this form the marginal willingness to pay is measurable also outside optimum, and so this open optimization problem has a clear advantage as compared with the restricted optimization in Eq. (1).*

Setting values  $\alpha = 0.7$ ,  $M = 100$  for the constants, we can present the demand and the marginal willingness to pay relations in this example with two values for  $P_1$ :  $P_{10} = 10$  and  $P_{11} = 20$ , see Fig. 1. The demand relation is graphed in the form of inverse demand. The demand relation stays constant while the marginal willingness to pay relation moves with the price change so that the two curves cross at current price. Thus only in a border optimum, marginal willingness to pay equals with Marshallian demand function.

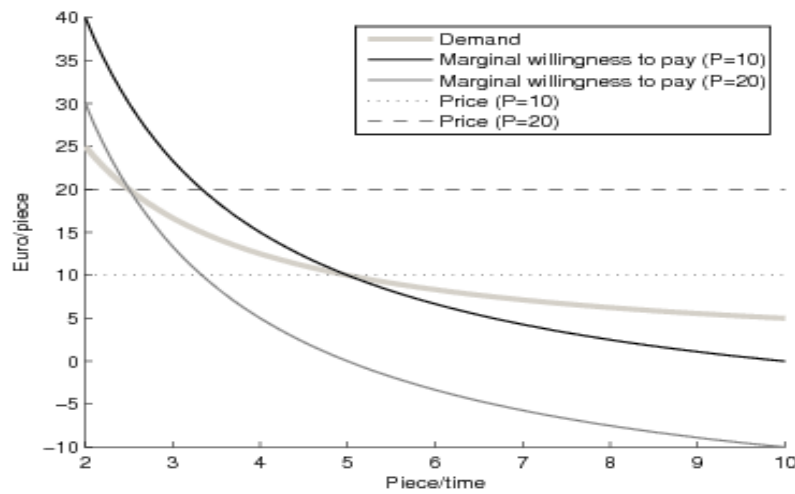


Figure 1. Demand and two marginal willingness to pay functions.

#### 4. Measuring changes in consumer welfare

Although indirect utility function  $v(\mathbf{P}, M)$  defines a complete order for utility levels of a consumer, its unit *utility/time* prohibits its empirical measuring. Due to this, welfare changes between utility levels have been measured by applying the duality theory of optimisation (see Eq. (8)) with expenditure function  $e(\mathbf{P}, v(\mathbf{P}, M))$  called the *money metric indirect utility function* (MM), see e.g., Ref. [12]: pp. 81-82. MM measures the minimum expenditures yielding a given utility level for the consumer. The change in welfare due to price change  $P_0 \rightarrow P_1$  has then been measured by Equivalent (EV) or Compensating Variation (CV) as

$$EV(\mathbf{P}_0, \mathbf{P}_1, u_1) = e(\mathbf{P}_0, u_1) - e(\mathbf{P}_1, u_1), \quad CV(\mathbf{P}_0, \mathbf{P}_1, u_0) = e(\mathbf{P}_0, u_0) - e(\mathbf{P}_1, u_0),$$

$$u_0 = v(\mathbf{P}_0, M), \quad u_1 = v(\mathbf{P}_1, M), \quad e(\mathbf{P}_0, u_0) = e(\mathbf{P}_1, u_1) = M.$$

The problems in using EV and CV are that they measure welfare changes (or minimum expenditures giving a fixed utility) only between optimum points, and they assume the utility function of the consumer to be known. In EV the utility is fixed at level  $u_1$ , and in CV at  $u_0$ , [12]: p. 83.

Next we study Example 1 and assume that  $P_2 = 3$ ,  $M = 1000$ ,  $\alpha = 0.7$  stay fixed and  $P_1$  changes so that  $\mathbf{P}_0 = (P_1, P_2) = (7, 3)$  and  $\mathbf{P}_1 = (P_1, P_2) = (3, 3)$ . The initial Marshallian demand of good 1 with  $P_1 = 7$  is  $X_{1M} = 100$  and  $P_1 = 3$  corresponds to  $X_{1M} = 233.33$ , while with  $P_2 = 3$ ,  $X_{2M} = 100$  is fixed  $\forall P_1$ . Next we calculate the change in welfare of the consumer by using the money metric utility function in Example 1 with  $B = 1$ . Now  $U(P_1 = 7, P_2 = 3) = u_0 = 100$ ,  $U(P_1 = 3, P_2 = 3) = u_1 = 180.96$ , and

$$EV(\mathbf{P}_0, \mathbf{P}_1, u_1) = 809.60, \quad CV(\mathbf{P}_0, \mathbf{P}_1, u_0) = 447.39. \quad (24)$$

Due to the decrease in  $P_1$ , the consumer's minimum expenditures giving utility level  $u_1$  decrease by 809.60 (€/time) (EV), and the minimum expenditures giving utility level  $u_0$  decrease by 447.39 (€/time) (CV).

CV has often been interpreted as the willingness to pay, and it has been used in calculating the consumer surplus, see e.g., Refs. [12]: Chapter 3, [13]: Chapter 3. The use of CV in evaluating a consumer's (marginal) willingness to pay for a good is dubious, however, since it measures the difference between minimum expenditures giving the initial level of utility at different prices. CV allows changes in flows of consumption for all goods, and thus it gives at best an indirect estimate of the money a consumer is willing to pay (or requires in compensation) for the price change of a good.

Let us compare, for example, the value 447.39 for measure CV when  $P_1$  changes from 7 to 3. The marginal willingness to pay (MWP) for good 1 given in Eq. (23) gives:  $P_1 = 7, X_1 = 90: MWP = 9.6$ ,  $P_1 = 7, X_1 = 95: MWP = 8.2$ ,  $P_1 = 3, X_1 = 90: MWP = 18.9$ ,  $P_1 = 3, X_1 = 95: MWP = 17.6$ . Thus our model measures a consumer's marginal willingness to pay for a good directly at all consumption levels, and it does not require the assumption of a price change.

Next we compare the Marshallian demand and the marginal willingness to pay functions in measuring a consumer's welfare by using the concept of consumer surplus. We calculate the consumer surplus assuming  $X_{10} < X_1^*$  and keeping  $P_2, M$  fixed. Using either the inverse demand function in Eq. (22) or the marginal willingness to pay function in

Eq. (23), we get the following two measures for the change in the consumer's surplus from the consumption of good 1:

$$\phi_1 = \int_{X_{10}}^{X_1^*} \left( \frac{\alpha M}{X_1} - P_1 \right) dX_1 = \alpha M \ln \left( \frac{X_1^*}{X_{10}} \right) - P_1 (X_1^* - X_{10}), \quad (25)$$

$$\phi_2 = \int_{X_{10}}^{X_1^*} \left( \frac{\alpha}{1-\alpha} \left( \frac{M}{X_1} - P_1 \right) - P_1 \right) dX_1 = \frac{\alpha M}{1-\alpha} \ln \left( \frac{X_1^*}{X_{10}} \right) - \frac{P_1 (X_1^* - X_{10})}{1-\alpha}, \quad (26)$$

where  $\ln$  is the natural logarithm. To get these measures comparable with those in Eq. (24), we assume  $M = 1000$ ,  $\alpha = 0.7$ , and  $X_{10} = 100$ . With two values for  $P_1$ , we get

$$\begin{aligned} A: X_1^* = 233.33 (P_1 = 3): \quad \phi_1 &= 193.11, & \phi_2 &= 643.70, \\ B: X_1^* = 140 (P_1 = 5): \quad \phi_1 &= 115.53, & \phi_2 &= 385.10. \end{aligned}$$

By comparing these two measures we see that in both cases  $\phi_1 < \phi_2$ . On the other hand, comparing case *A* with that in Eq. (24) we see that  $\phi_1$  (Marshall) is less than CV while  $\phi_2$  (MWP) is between CV and EV. Thus the Marshallian demand function essentially deviates from the Hicksian in calculating changes in welfare, and MWP function gives values more in line with those of EV and CV. Another advantage of applying the marginal willingness to pay in measuring welfare is that it can be measured by making a consumer survey.

## 5. A new theory of a consumer

In Lagrangian functions (1) and (5),  $\lambda$  with unit *utility*/€ and  $\mu$  with unit €/*utility* make utility and money additive. On the other hand, in a border optimum there exists transformation equation  $U(\mathbf{X}_M) = v(\mathbf{P}, M)$  between utility and money, see Eq. (8). However, exact functional forms for these two ways of transforming utility and money are obtained only in a border optimum.

The theory of expected utility, introduced by Bernoulli [26] and further developed by von Neumann and Morgenstern [27], states that for a risk-averse consumer, the conversion function between utility and money is concave. Thus the conversion function of a risk-averse consumer between utility and money is of the form  $U = f(M)$ ,  $f'(M) > 0$ ,  $f''(M) < 0$ , where  $M$  is the consumer's income and  $U$  his/her level of utility.

By looking at  $\lambda^*$  and  $\mu^*$  in Eq. (8) we see, however, that in the optimum the two conversion functions do not depend on the consumer's

income. On the other hand,  $\lambda^*$  negatively (and  $\mu^*$  positively) depends on the two prices; the marginal utility of money is thus the smaller the higher the prices. Hyperinflation in real economies (e.g. Germany in 1923, Hungary in 1946, and Zimbabwe in 2008) is an example of this. Relation  $U(\mathbf{X}_M) = v(\mathbf{P}, M)$  between  $U$  and  $M$  in Eq. (8) is, instead, linear. Thus the conversion rules between utility and money in the neoclassical framework are in conflict with expected utility theory [26, 27].

The open optimization problem in Eq. (12), however, gives the required non-linear relationship:

$$\frac{\partial U}{\partial M} = \frac{1}{P_2} \frac{\partial f(X_1, X_2)}{\partial X_2} > 0,$$

$$\frac{\partial^2 U}{\partial M^2} = \frac{1}{P_2^2} \frac{\partial^2 f(X_1, X_2)}{\partial X_2^2} < 0 \quad \text{iff} \quad \frac{\partial^2 f(X_1, X_2)}{\partial X_2^2} < 0.$$

The problem with this utility function is, however, that the relation between utility function and willingness to pay function, as well as the willingness to pay function and marginal willingness to pay functions, is complicated. The marginal willingness to pay for good 1 in Example 2, e.g., is given in Eq. (23). Integrating this we get the willingness to pay function for good 1 as:

$$H(X_1, M, P_1) = \int_{X_{10}}^{X_1} \frac{\partial H}{\partial X_1} dX_1 = \left( \frac{\alpha}{1-\alpha} \right) \left[ M \ln \left( \frac{X_1}{X_{10}} \right) - P_1 (X_1 - X_{10}) \right]. \quad (27)$$

The problem with  $H$  in Eq. (27) is that it is not general enough that we could derive from it marginal willingness to pay functions for goods 1 and 2. We can though substitute  $X_1$  from the utility function by the budget equation to get the willingness to pay function for good 2 as in Eq. (27). However, the assumption of binding budget equation essentially distorts the modeling principles of consumer behavior, and thus we reformulate the theory as follows. We modify the Lagrangian utility maximisation problem of a consumer as

$$\max_{\mathbf{X}} \Phi = \eta(M)U(\mathbf{X}) + M - \sum_{i=1}^n P_i X_i, \quad (28)$$

where  $\eta(M)$  with unit  $\text{€}/\text{utility}$  is not the inverse of the Lagrangian multiplier  $\lambda$  in Eq. (1), but a consumer specific function that converts this consumer's utility from consumption into monetary units. According to the neoclassical theory, in a border optimum holds  $\partial U / \partial M = \partial v / \partial M = \lambda^* > 0$ , and from Eq. (8) we see that in this specific case  $\partial^2 U / \partial M^2 = \partial^2 v / \partial M^2 = 0$ . Now, because  $\eta(M)$  resembles  $1/\lambda$  in Eq. (1), we derive

the properties for function  $\eta(M)$  according to the following approximation:  
 $\eta(M) \approx 1/\lambda(M)$ ;

$$\eta(M) \approx \frac{1}{\lambda(M)} > 0, \quad \eta'(M) \approx -\frac{1}{\lambda^2} \frac{\partial \lambda}{\partial M} > 0. \quad (29)$$

The results in Eq. (29) are caused by the assumption of decreasing marginal utility of money,  $\partial^2 U / \partial M^2 = \partial \lambda / \partial M < 0$ . Thus with higher income a consumer is willing to pay more for a fixed level of consumption because extra income is less valuable<sup>8</sup>.

Lagrangian (28) measures the net utility  $\Phi$  of the consumer expressed in monetary units. The target function  $\Phi$  of the consumer consists of his/her own valuation of his/her current consumption in monetary units  $\eta(M)U(\mathbf{X})$  added by net savings in terms of income minus expenditures. The term  $H(\mathbf{X}, M) \equiv \eta(M)U(\mathbf{X})$  with unit  $\text{€}/\text{time}$  can be interpreted as the *willingness to pay of the consumer for his/her consumption flow vector  $\mathbf{X}$* , or the consumer's monetary valuation of his/her consumption flow vector  $\mathbf{X}$ .

The difference between Eqs. (1) and (28) can be described as follows. The marginal willingness to pay for good 1 with budget constraint is given in Eq. (23). This deviates from the following general formulation for the marginal willingness to pay for good 1 obtained from Eq. (28)

$$\frac{\partial H}{\partial X_1} = \eta(M) \frac{\partial U}{\partial X_1} = \eta(M) \alpha \left( \frac{X_1}{X_2} \right)^{\alpha-1}.$$

There exists reasons for Eq. (28) to be a more exact expression of consumer behaviour than the neoclassical Eq. (1). 1) It makes the calculation of marginal willingnesses to pay for every good easy and intuitive. 2) It is an open optimization problem as in the theory of a firm. The symmetry between the models for a firm and a consumer has its uses e.g. in dynamizing the theory, and in developing an analytic microeconomic theory similar to analytical mechanics in physics. 3) Eq. (28) is general while budget constraint represents a special case. It is also irrational to think that a consumer always spends all his/her money.

The income of a consumer at a time unit consists of labour income and interest revenues or costs, depending on whether he/she has positive or

---

<sup>8</sup> Function  $\eta$  can be generalised by including time in it because the utility-money conversion rate of a consumer changes as he/she gets older. As the preferences of humans also change in time, utility function, strictly speaking, has a similar time dependency. In relatively short time intervals, however, one can quite safely assume that functions  $\eta$  and  $U$  do not change significantly in (a short) time.

negative net wealth. Positive net wealth  $W > 0$  with unit € gives interest earnings with interest rate<sup>9</sup>  $r$  having unit  $1/time$ , and negative net wealth  $W < 0$  causes interest payments. Thus  $M = M_0 + rW$ , where  $M_0$  is labour income and  $rW$  capital gains (if  $W > 0$ ) or costs (if  $W < 0$ ) during a time unit; both these income components have unit €/time. Net savings  $M - \sum_{i=1}^n P_i X_i$  may be positive or negative, and they affect the future wealth of the consumer.

Now, no budget constraint exists in the optimisation problem of Eq. (28), and the consumer may save money or consume with credit. From Eq. (28) we get

$$\frac{\partial \Phi}{\partial X_i} = 0 \Leftrightarrow \eta(M) \frac{\partial U}{\partial X_i} = P_i, \quad i = 1, \dots, n, \quad \frac{\partial \Phi}{\partial M} = \eta'(M)U(X) + 1 > 0. \quad (30)$$

The sufficient conditions for maximum are

$$\frac{\partial^2 \Phi}{\partial X_i^2} = \frac{\partial^2 H}{\partial X_i^2} = \eta(M) \frac{\partial^2 U}{\partial X_i^2} < 0, \quad \forall i, \quad (31)$$

where  $\partial^2 U / \partial X_i^2 < 0$  results from the law of decreasing marginal utility. Thus a rational consumer does not consume infinite amounts of any good, even if the consumer's credit or income were unlimited. The optimum conditions in Eq. (30) coincide with those in Eq. (2), if  $\eta = 1/\lambda$  and the solution of the Lagrangian (28) obeys  $M \geq \sum_{i=1}^n P_i X_i$ . However, the essential difference between Eq. (2) and Eq. (30) is that in the former, the marginal willingness to pay for good  $i$   $(\partial U / \partial X_i) / \lambda$  is not measurable outside optimum while in the latter,  $\eta(M)(\partial U / \partial X_i)$  is measurable everywhere. Result  $\partial \Phi / \partial M$  shows that income affects the consumer's net monetary utility not only directly but also indirectly via altering his/her willingness to pay for consumption.

While Lagrangian (28) allows savings and crediting, it does not take into account possible credit limit or paying back possible loans. These extensions will be considered in future studies.

Differentiating Eq. (28) (with fixed  $M$ ) yields

$$d\Phi = \sum_{i=1}^n \frac{\partial \Phi}{\partial X_i} dX_i = \sum_{i=1}^n \left( \eta \frac{\partial U(X)}{\partial X_i} - P_i \right) dX_i. \quad (32)$$

The consumer surplus  $\phi$  with unit €/time is then obtained as

---

<sup>9</sup> For simplicity, the same interest rate is assumed for loans and investments. The unit of interest rate (the growth rate of invested capital) results from dividing the flow of interest revenues with unit €/time by the invested capital with unit €, see Ref. [23].

$$\begin{aligned}\phi &= \int d\Phi = \Phi - M = \\ \sum_{i=1}^n \int_0^{X_i} \left( \eta \frac{\partial U}{\partial X_i} - P_i \right) dX_i &= \eta U(\mathbf{X}) - \sum_{i=1}^n P_i X_i,\end{aligned}\quad (33)$$

where  $\phi(0) = 0$  is the natural initial condition and the willingness to pay of the consumer for  $\mathbf{X}$  is  $H(\mathbf{X}, M) \equiv \eta(M)U(\mathbf{X})$  with unit  $\text{€}/\text{time}$ .<sup>10</sup> The initial condition for the willingness to pay emerges from the rationality of the consumer: the consumer pays nothing for nothing, i.e.,  $\eta(M)U(0) = H(0, M) = 0$ , and  $\partial H / \partial X_i = \eta(M) \partial U / \partial X_i \geq P_i$  for  $0 \leq X_i \leq X_i^*$ , where by  $X_i^*$  is denoted the optimal consumption. The unit of marginal willingness to pay for good  $i$  is  $\text{€}/\text{piece}_i$ , and the relation between the maximal net monetary utility of a consumer and the consumer's surplus is:  $\Phi(\mathbf{X}) = \phi(\mathbf{X}) + M$ .

The form of the consumer surplus in Eq. (33) is intuitive since it is the difference between the willingness to pay of the consumer  $H(\mathbf{X}, M)$  and his/her expenditures  $\sum_{i=1}^n P_i X_i$  for the consumption vector  $\mathbf{X}$ . The consumer surplus of a particular good is treated similarly:

$$\phi_i = \int_0^{X_i} \left( \eta \frac{\partial U}{\partial X_i} - P_i \right) dX_i = \int_0^{X_i} \eta \frac{\partial U}{\partial X_i} dX_i - P_i X_i.\quad (34)$$

Eq. (34) is consistent with the consumer surplus presentations in economics textbooks, see e.g. Refs. [5, 10].

**Example 3.** Assume the utility function as in Example 1. The willingness to pay function is then

$$H(\mathbf{X}, M) = \eta(M) B X_1^\alpha X_2^{1-\alpha}.$$

The consumer's marginal willingness to pay for the two goods are

$$\frac{\partial H}{\partial X_1} = \alpha \eta(M) B \left( \frac{X_2}{X_1} \right)^{1-\alpha}, \quad \frac{\partial H}{\partial X_2} = (1 - \alpha) \eta(M) B \left( \frac{X_1}{X_2} \right)^\alpha,\quad (35)$$

and the optimum conditions are

$$\begin{aligned}\frac{\partial H}{\partial X_1} = P_1 &\Leftrightarrow \alpha \eta(M) B \left( \frac{X_2}{X_1} \right)^{1-\alpha} = P_1, \\ \frac{\partial H}{\partial X_2} = P_2 &\Leftrightarrow (1 - \alpha) \eta(M) B \left( \frac{X_1}{X_2} \right)^\alpha = P_2.\end{aligned}$$

The condition that holds in every optimum of the consumer is

$$\frac{P_1 X_1}{\alpha} = \frac{P_2 X_2}{1-\alpha},\quad (36)$$

---

<sup>10</sup> In the differentiation of Lagrangian (25), constant  $M$  vanishes but it is obtained by integrating the surplus of the consumer.

and in the case the budget equation  $M = P_1X_1 + P_2X_2$  is binding, we get the neoclassical optimum:

$$X_{1M} = \frac{\alpha M}{P_1}, \quad X_{2M} = \frac{(1-\alpha)M}{P_2}. \quad (37)$$

One should notice that if the budget equation holds, function  $\eta$  does not affect the optimum flows of consumption in Eq. (37), nor the marginal rate of substitution between the two goods in Eq. (36), but it does affect the consumer's marginal willingness to pay for the two goods in Eq. (35).

## 6. Dynamising the theory of a consumer

As compared with the neoclassical theory, the final advantage of our framework is that dynamising the theory is straightforward. The neoclassical consumer theory has been presented in dynamic form e.g. in Refs. [28, 29, 30]. However, it is shown in Ref. [31] that these studies model the dynamics of the money a consumer allocates in consumption with time, while the static theory explains a consumer's real consumption of different goods. Thus the two theories model different quantities and the former are not dynamisations of the latter.

A utility-seeking consumer increases his/her flow of consumption of a good when his/her marginal willingness to pay for the good exceeds its price, and vice versa. Let  $X_i$  be a function of time  $t$ , and  $\dot{X}_i \equiv dX_i/dt$ . Dynamic consumer behaviour can then be expressed as

$$m_i \dot{X}_i(t) = \eta(M) \frac{\partial U}{\partial X_i} - P_i, \quad \forall i, \quad (38)$$

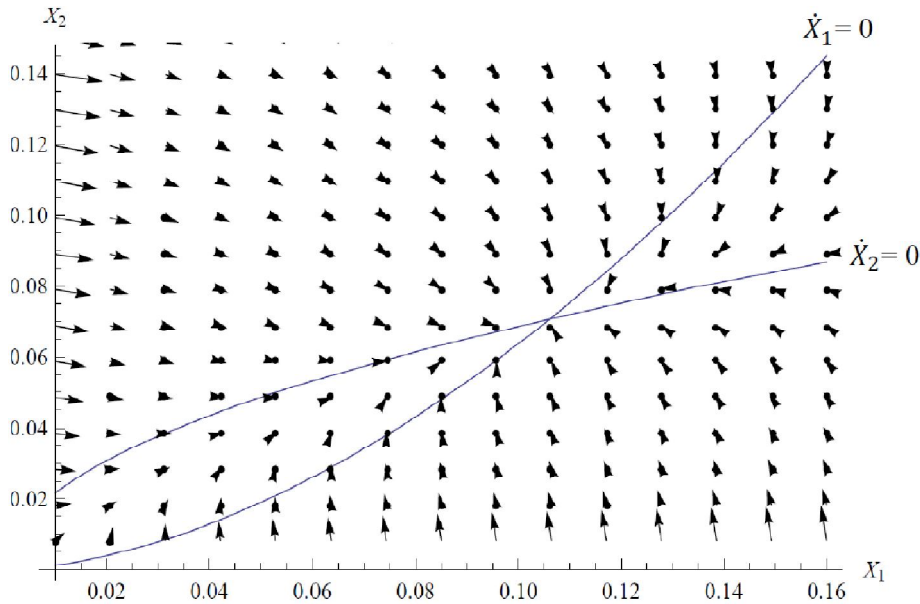
where constants  $m_i > 0$  with units  $\text{€} \times \text{time}^2/\text{piece}_i^2$  make the equations well-defined in dimensions. Now,  $\dot{X}_i(t)$  with unit  $\text{piece}_i/\text{time}^2$  corresponds to the acceleration of consumption of good  $i$ , and imitating Newtonian mechanics we interpret quantity  $\eta \partial U / \partial X_i - P_i$  as the "force" acting upon the consumption of good  $i$  of this consumer, and  $m_i$  as the inertial "mass" of this consumer's consumption of good  $i$ . Quantity  $m_i$  measures the sensitivity of  $\dot{X}_i(t)$  with respect to the force  $\eta \partial U / \partial X_i - P_i$ . Eq. (38) shows that  $\dot{X}_i(t) > 0$  if  $\eta \partial U / \partial X_i > P_i$ , and vice versa, and state  $\dot{X}_i(t) = 0, \forall i$  — the neoclassical equilibrium — corresponds to zero force  $\eta(\partial U / \partial X_i) = P_i, \forall i$ , if  $\eta = 1/\lambda$ .

**Example 4.** Let the utility function of a consumer in a two-good case be as in Example 1 so that  $1 - \alpha$  is changed to  $0 < \beta < 1$ ; thus  $\alpha + \beta = 1$  may not hold any more. Eqs. (38) become then in the form

$$m_1 \dot{X}_1(t) = \alpha \eta(M) B X_1^{\alpha-1} X_2^\beta - P_1 \quad (39)$$

$$m_2 \dot{X}_2(t) = \beta \eta(M) B X_1^\alpha X_2^{\beta-1} - P_2. \quad (40)$$

Due to the non-linearity of the system, we do not try to solve it analytically but only study its behaviour by phase diagrams. Two possible cases occur depending on whether the slope of line  $\dot{X}_1(t) = 0$  is steeper or gentler than that of line  $\dot{X}_2(t) = 0$ . The stable case occurs if line  $\dot{X}_1(t) = 0$  is steeper, see Fig. 2. In this case, the consumer sooner or later ends up into his/her neoclassical equilibrium. Thus our model explains how a consumer finds his/her optimum after a price of an income change.



**Figure 2.** A stable case.

If line  $\dot{X}_2(t) = 0$  is steeper, however, the equilibrium is a saddle and the consumptions of both goods either diminish to zero or increase without limit, see Fig. 3. In both figures,  $\eta = 20, B = 0.5, P_1 = 5, P_2 = 10, M = 1000$ , and in Fig. 2,  $\alpha = 0.3, \beta = 0.4$  and in Fig. 3,  $\alpha = 0.5, \beta = 0.7$ . One should notice that the marginal utilities of both goods are decreasing in both cases.

The instability in Fig. 3 results from increasing returns to scale in consumption, i.e.,  $\alpha + \beta > 1$ . The flow of consumption of good 1

positively affects the marginal willingness to pay for good 2, and vice versa. If the initial consumptions of both goods are high, this keeps the marginal willingness to pay values higher than the prices of the goods even though the marginal utilities of both goods are decreasing. Thus both consumptions increase without limit. Similarly, low current consumptions of the two goods keep the marginal willingness to pay values smaller than the prices of the goods, which decreases both consumptions. These results originate from the multiplicative form of the utility function, and if this is not realistic, the form of the utility function should be changed.

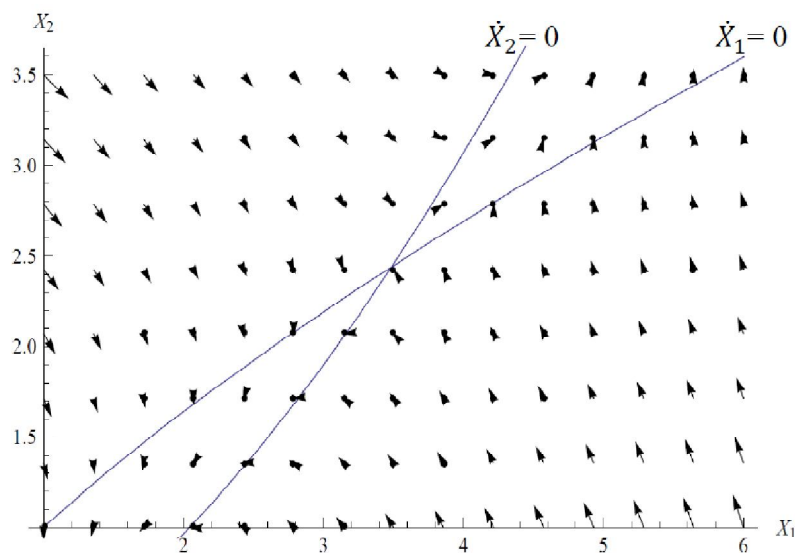


Figure 3. A non-stable case.

This dynamisation of consumer theory will be treated more explicitly in future studies. It is presented here only to show how the framework can be dynamised so that time-dependent prices, income, and preferences are natural elements in it. These time-dependencies are impossible in the neo-classical framework.

## 7. Relations between the main concepts in consumer theory

The relations between the applied concepts in consumer theory are: Utility function  $U(\mathbf{X})$  with unit *utility/time* measures the "satisfaction" of a consumer from consuming all  $n$  goods simultaneously at the consumption flow vector  $\mathbf{X}$ , and  $\partial U / \partial X_j$  measures the marginal utility with unit *utility/piece<sub>j</sub>* at  $\mathbf{X}$ . By integrating  $\int (\partial U / \partial X_j) dX_j = U_j(X_j) + U_{j0}$  we get the partial utility function  $U_j$  for good  $j$  at fixed  $X_{\neq j}$  with unit

*utility/time*;  $U_{j0}$  is the constant of integration. Similarly, function  $H(\mathbf{X}, M)$  with unit  $\text{€}/\text{time}$  measures the consumer's willingness to pay for the consumption flow vector  $\mathbf{X}$ , and  $\partial H / \partial X_j$  with unit  $\text{€}/\text{piece}_j$  measures the consumer's marginal willingness to pay for good  $j$  at  $\mathbf{X}$ . By integrating  $\int (\partial H / \partial X_j) dX_j = H_j(X_j, M) + H_{j0}$ ,  $H_{j0}$  constant, we get the partial willingness to pay function  $H_j$  for good  $j$  at fixed  $X_{\neq j}$  with unit  $\text{€}/\text{time}$ .

The partial willingness to pay function  $H_j$  can be used in studies of consumer behaviour related to a certain good. For example, one can solve the form of the marginal willingness to pay function for good  $j$  by asking "*How much would you pay for one unit of good  $j$  when you have consumed  $x$  units of the good in a time unit?*". By integrating this answer, one can deduce the partial willingness to pay function for good  $j$  for the consumer, and knowing the price of good  $j$  (and the consumption of the consumer), one can calculate the consumer surplus for this consumer from good  $j$ . However, one can neither solve the consumer's willingness to pay function for all goods, nor his/her utility function by studying only the behaviour related to good  $j$ .

Real consumers tend to intuitively understand their marginal willingness to pay  $\partial H / \partial X_i$  for good  $i$  because in their consumption decision, they must compare it with price  $P_i$ . On the other hand, real consumers hardly know their marginal utilities of goods or their utility function. However, they may be able to give a monetary value for their consumer surplus from good  $i$  via their marginal willingness to pay. Thus marginal willingness to pay is a more fundamental concept in modelling consumer behaviour than marginal utility.

## **8. Discussion**

The theory of a consumer has so far not been expressed in a form where the concepts of utility, marginal utility, demand, willingness to pay, and marginal willingness to pay together with their mutual relations had been properly defined. In the neo-classical optimum, a consumer's marginal willingness to pay equals with the price for every good, i.e.  $(\partial U / \partial X_i) / \lambda = P_i$ . If defined in this way, however, the marginal willingness to pay cannot be calculated outside optimum because there  $\lambda$  is not measurable. Here we tried to fill these shortcomings by giving exact definitions for the key terms and by defining the marginal willingness to pay in a new way.

Concerning the empirical testing of the neoclassical consumer theory, utility function and marginal utilities are difficult, maybe impossible to be measured directly. In our theory, however, all necessary variables are directly measurable, i.e., the prices of goods, the income (or money allocated for consumption) of a consumer, and the consumer's willingness and marginal willingness to pay for goods. Thus, our theory serves as a theoretical base for empirical studies of consumer behaviour that can be directly tested. Moreover, our theory allows time-dependent variables, too, that are crucial in dynamising the theory.

### Acknowledgements

The authors are indebted to Anna Dannenberg for critical reading of the manuscript. Alia Asha Dannenberg thanks Jenny and Antti Wihuri Foundation and Emil Aaltonen Foundation for the grants that enabled her work.

### REFERENCES

- [1] J. Hicks, *Causality in Economics* (Basic Books, New York, 1979).
- [2] B. De Uriarte, *On the free will of rational agents in neoclassical economics*, *Journal of Post Keynesian Economics* **12**(4), 605-617 (1990).
- [3] G. Gigerenzer, R. Selten, *Bounded Rationality* (MIT Press, Cambridge, 2002).
- [4] D. M. Hausman, *Philosophy of Economics*, in *The Stanford Encyclopedia of Philosophy* (Spring 2013 Edition), edited by E. N. Zalta, (URL = <http://plato.stanford.edu/archives/spr2013/entries/economics/>, 2013).
- [5] J. V. Henderson, W. Poole, *Principles of economics* (D. C. Heath and Company, Toronto, 1991).
- [6] T. A. Cameron, M. D. James, *Estimating willingness to pay from survey data: An alternative pre-test-market evaluation procedure*, *Journal of Marketing Research* **24**(11), 389-395 (1987).
- [7] B. O'Brien, J. L. Viramontes, *Willingness to Pay: A Valid and Reliable Measure of Health State Preference?* *Medical Decision Making* **14**, 289-297 (1994).
- [8] K. Wertenbroch, B. Skiera, *Measuring Consumers' Willingness to Pay at the Point of Purchase*, *Journal of Marketing Research* **39**(2), 228-241 (2002).
- [9] J. L. Lusk, D. Hudson, *Willingness-to-Pay Estimates and Their Relevance to Agribusiness Decision Making*, *Applied Economic Perspectives and Policy* **26**(2), 152-169 (2004).
- [10] M. Parkin, D. King, *Economics*, 2nd edn. (Addison-Wesley Publishing, Workingham, 1995).
- [11] C. Liao, H. Önal, M.-H. Chen, *Average shadow price and equilibrium price: A case study of tradable pollution permit markets*, *European Journal of Operational Research* **196**, 1207-1213 (2009).

- [12] A. Mas-Colell, M. D. Whinston, J. R. Green, *Microeconomic Theory* (Oxford University Press, New York, 1995).
- [13] H. Gravelle, R. Rees, *Microeconomics*, 3rd edn. (Prentice Hall, London, 2004).
- [14] S. Rosen, *Valuing health risk*, *The American Economic Review* **71**(2), 241-245 (1981).
- [15] S. K. Swallow, R. A. Sedjo, *Eco-labeling consequences in general equilibrium: A graphical assessment*, *Land Economics* **76**(1), 28-36 (2000).
- [16] T. B. Björner, L. G. Hansen, C. S. Russell, *Environmental labeling and consumers' choice: an empirical analysis of the effect of the Nordic Swan*, *Journal of Environmental Economics and Management* **47**(3), 411-434 (2004).
- [17] A. A Dragulescu, V. M. Yakovenko, *Statistical mechanics of money*, *The European Physical Journal B* **17**, 723-729 (2000).
- [18] F. V. Kusmartsev, *Statistical mechanics of economics I*, *Physics Letters A* **375**, 966-973 (2011).
- [19] A. Lewbel, *Exact aggregation and a representative consumer*, *The Quarterly Journal of Economics* **104**(3), 621-633 (1989).
- [20] A. P. Kirman, *Whom or what does the representative individual represent?*, *Journal of Economic Perspectives* **6**, 117-136 (1992).
- [21] M. Gallegati, A. P. Kirman, *Beyond the Representative Agent* (Edward Elgar, NH, 1999).
- [22] Y. Chang, S.-B. Kim, F. Schorfheide, *Labor-Market Heterogeneity, Aggregation, and Policy (In)variance of DSGE Model Parameters*, *Journal of the European Economic Association* **11**, 193-220 (2013).
- [23] F. J. de Jong, *Dimensional Analysis for Economists* (North-Holland Publishing Company, Amsterdam, 1967).
- [24] J. M. Perloff, *Microeconomics*, 6th edn. (Pearson, Prentice Hall, 2012).
- [25] M. Estola, *Newtonian Microeconomics: A Dynamic Extension to Neoclassical Micro Theory* (Palgrave Macmillan, 2017).
- [26] D. Bernoulli, *Exposition of a New Theory on the Measurement of Risk (1738)*; *Translated by Dr. Louise Sommer*, *Econometrica* **22**(1), 22-36 (1954).
- [27] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, 3rd edn. (Princeton University Press, Princeton, 1953).
- [28] F. P. Ramsey, *A Mathematical Theory of Saving*, *Economic Journal* **38**(December), 543-559 (1928); Reprinted in Stiglitz and Uzawa (1969).
- [29] D. Cass, *Optimum Growth in an Aggregative Model of Capital Accumulation*, *Review of Economic Studies* **32**(July), 233-240 (1965).
- [30] T. C. Koopmans, *On the Concept of Optimal Economic Growth*, in *The Economic Approach of Development Planning* (North Holland, Amsterdam, 1966).
- [31] M. Estola, *Consistent and Inconsistent Ways to Dynamize the Neo-classical Theory*, *Hyperion International Journal of Econophysics & New Economy* **6**(1) (2013).