A TRAIL BETWEEN RIEMANN HYPOTHESIS AND THE FOUNTS OF CURRENCY

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Abstract. *The epistemological status of the currency is an open question that has been debated for centuries including in its metaphysical aspects. By reducing the currency to a mere factor of dynamics equilibrium between fungible goods, the classical approach seems unable to grasp the complexity of the being at world of the money. This paper proposes a deepening of the assumption in which the currency is, – within the complex system of an economical analytics, incomplete in its nature –, the non-causal part of a categorical exchange herein mathematically defined. Whereas this dynamics is, in accordance with universal theorem of Voronin, associated with the Riemann Zeta function - as a trace of the exponential operator upon the set of natural number – given at a scale s, its dual, namely the currency, emerges under a (1 – s) zeta-control, as a complementary factor required to supply a steady state to the naturally inceptive s-system. This point of view is based on a pioneering approach of the Riemann Hypothesis, whose the categorical methods, herein analyzed in detail, are extended for taking into account a time-singularity, caused by the scaling, and appearing at the boundary of the set of complex time. The dualism is then tuned by the functional properties of zeta function. This analysis shows why, implicitly, the Riemann Hypothesis applied without cautions in the physics of the complex systems, suggests a reduction of the currency to a mere commodity whose dynamics then becomes stochastic and in fine strictly determined on a basis of frozen economic states (equivalent to the zeros of the zeta function) which obviously cannot exist. The only freedom of this economical fantasy is related to a natural uncertainty. The main characteristics of the capitalism disappear, identically the irreversibility and with it all the dreams into the future. A careful analysis of the Riemann Hypothesis new approach highlights the mismatching between mathematical and econophysics models and opens the way for a conceptual breakthrough that nevertheless stays based on zeta function.*

Index terms: *Currency, Econophysics, Category Theory, Zeta function, Riemann Hypothesis.*

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I. Introduction

Initially considered within a theological approach, the rule that legitimates the function of the currency in the economic frame stays yet without clear answer [1, 2, 3, 4, 5, 6]. Within a more general framework, the issue of role of the money in economy is only one example of the questions open by the incompleteness that always characterizes Complex Systems. The scientist is indeed compelled by his inability to apply a reductionist approach, because this approach is based, in particular, upon a clear separation, herein impossible, between the subject and the object [7, 8]. Such is the case, for auto-referential organizations like the living systems, creative process, the dynamics of social structures (among them the economics systems) etc. In the latter context, the status of the currency is both a theoretical enigma and a singular historical experience. The currency carries with it all the paradoxes of the entanglement of questions that have to be addressed when the Cartesian principles become inefficient. It is unnecessary herein to return to paradoxes substantiated by many academic controversies and numerous economics works [3, 4, 2]. These works mainly lead to the following issue: is the economy a true science? The lack of a consensus about mathematical models [3, 9]and the extensive use of statistics to deal with monetary uncertainties is a clinical sign of our inability to properly identify the position of the currency in an economical framework distinct of a mere gambling.

We already issued elsewhere [8] the assumption that the Riemann zeta function and in particular the conjecture concerning the distribution of its non-trivial zeros of this function, are likely to enlighten the currency position in the economy. We assert that the analysis of the properties of the zeta function provides an approach able to shed light onto the fundamental incompleteness that would characterize when currency is lacking, the economical exchanges; incompleteness precisely "traced" by the function of Riemann. In this context, like the musical score with respect to the interpretation, the money founds the "existential project" of any economical object; it is the causality of this living object; it is its capability to fulfill the fundamental incompleteness of the economy, classified as a Complex System. We would return to this question not only by using a categorical approach of the Riemann Hypothesis, but mainly by showing that understanding the incompleteness factors not found solely in the distribution of zeros, – namely in the application of the Poincaré principle which claims that behavior can be depicted by this set –, but in a widest parametrical vision which takes into account the functional relationships that characterizes the zeta function [10, 11, 12, 13, 14]. This observation involves vast epistemological implications and social involvements not reducible to stochastics models even usually applied in economy [15] and in physics [16, 17, 18]

More generally the physical point of view on Riemann Hypothesis, shows that the analysis leads to assign to the "time" variable, – the parameter of any self-referential analytical functions –, a complex mathematical expression whose "irreversible characteristics" must be analyzed in detail. What is the meaning of the word "irreversibility" in the framework of holomorphic set of functions? Strangely, and by taking the problem by reverse, we will able to observe that the irreversible characteristics disappear when some conditions are fulfilled; for instance when the metric of the space of exchanges (considered as Riemann manifold) acquires integer dimensional values. It is obviously the case when the system is deterministic but it is also the case if the system is stochastique with Gaussian properties [15]. This remark involves prominent consequences for instance in the policy aiming to connect the monetary control and the policy of investments or conversely a policy of quantitative easing given to the banks with respect to the same quantitative easing offer to a set of consumers. We assert that these questions of the relationship between time irreversibility and "metric" of the commercial relationships are lighted by zeta Riemann function. That last offers presumably one of the most interesting tools whether the rebuilding an alternative model of economy becomes a social aim, for instance for facing a situation of economic stagnation.

Based on the work related to the categorical understanding of the analytical aspects of Riemann Hypothesis via the universal theorem of Voronin and the Bagchi lemma, (explicitly the self-similarity of the zeta function)and beyond the only relationships $\zeta(s)=0$ and the Riemann Hypothesis [19, 20], this note is intended to highlight, the central role of the functional relationship and the symmetries that typify the zeta function according to $\zeta(s)=F[\zeta(1-s)]$ relation which will suggest through a generalization of the Fourier transform, to attribute *in fine* an involution feature to the currency. With respect to the currency status, the analysis will propose an alternative to the sole probabilistic concerns, that last option suggesting implicitly and without any malicious will, the existence of a frozen society within an eternal-time thermalized by the chance and a will of infinite stability (namely a non-capitalistic structure). To illustrate this approach the paragraph II describes what we call a dynamic expon**a**ntial (with "a" and not "e"), [21, 22, 23]. Section III points to the role played by the symmetries of the zeta function in the completion process, the currency corresponding to a teleonomic extension of the causal dynamics of exchanges. Section IV summarizes (by giving a physical meaning) the state of mathematics and categorical studies already conducted by the authors in which the original approach of the Riemann Hypothesis is based on the formal closure of a couple of monoidal structures (multiplicative and additive build upon $\mathbb{N}$ the set of integers). That closure leads to highlight the role of the Riemann zeta function as a trace of an exponential operator upon $\mathbb{N}$. The closure involves the emergence of renormalization groups and of idempotent transformations, both functions highlighting the role of the infinite fix points sets either open (need of currency) or closed over itself (currency behaving as commodity). By generalizing the notion of "time" this section will emphasize the role of fiber-categories for understanding the need of enlargement (forcing, Kan extension) of the Complex Systems. This work unveils, through some arithmetical subtleties, the solution of some paradoxes and explains the deeper meaning of Voronin and Bagchi's theorems [24, 25, 26]. The section V, returns to the original concept of expon**a**ntielle. The analysis sheds a new light on the problem of irreversible time in physics, through the emphasis upon the requirement for labeling the set of integers $\mathbb{N}$, – far from the sole stochastic point of view and in the framework of a new epistemological aim –, the issues open by the concept of 'referential' that introduces then a new status of currency. The section VI, shows how the currency is the archetype of the methods based on Riemann zeta function, for completing what we suggest to categorize henceforth under the label of Zeta-Complex Systems (herein the economic system).

II. The concept of exponantial

Within the framework of classic analysis and algebra, it is well known that the logarithm function and its dual, the exponential function, guaranty the permutation between addition and multiplication. This property founds probably the role of similarities in physics [27]. We have to recall nevertheless that this permutation is not always justified. It requires mathematical conditions depicted for instance in detail by the theory of category (the theory of the arrows) [28, 29, 30, 31]. As it was highlighted by the authors – approaching, via to the computer sciences, the physical signification of the Riemann Hypothesis [8] –, in the general case of the use of monoid for representing irreversible transforms (transformation oriented of objects), these last denies intuitive relevance of any matching between addition (seen like constructive performance) and multiplication (seen like a capability of a partition of a set). The matching of the couple of monoids, product and co-product, requires an inversion of arrows and a meshing between monomorphisms and epimorphisms. It can then be proved that a renormalization group (and a self-similar algebraic structure) emerges if a closing is implemented. A self-similarity thus ensures asymptotically the mathematical rightfulness of the closure. In addition, it is known that the Fourier transform builds also a peculiar link between addition and multiplication performing implicitly a regularization (completion using test functions) of some singular distributions. For instance let us consider a straight line, the compactification which involves that $\mathbb{R}$ is extended to the complex $\mathbb{C}$ plane, is achieved by matching the infinite straight line parametrized by $in = i\omega\tau$, with a compact domain of the $\mathbb{C}$ by taking an outside singular point outside of the line, marked "0" and by implementing a geometric inversion which actualizes the infinity. This inversion leads to a half circle parametrized by a number "*n*" (Figure 1). That number can be implemented through the couple of natural numbers $n = \omega\tau$, an more precisely if a discretization is implemented in the context of the integers $\mathbb{N}$, n can be decomposed into primes p_i according to the well-known relationship $n = \prod_{i \in \mathbb{N}} (p_i)^{\tau_i}$, set that may be obviously split into two classes ω and τ distributed under different possible expressions. We shall return later to the meaning of this partition but let us notice immediately the following features: The Fourier transform [32] leads the birth of a semicircle in the complex plan whose equation is given by $Z = 1/[1 + i\omega\tau]$. The half circle is parameterized as is well known by $n = \omega\tau$ according to the first order dynamics equation leading the exponential time like relaxation $\varphi(t) \sim \exp(t/\tau)$. Let us observe that the time used herein is the time of the classical mechanics, namely the reversible parameterization of a continuous function. For diffusion differential equation, the relaxation process is based upon a half-power law $1/\sqrt{\pi t}$ whose Fourier transform is $1/\sqrt{i\omega\tau}$ or more complex forms always well-defined if additional factors are involved. The irreversibility carried by the relaxation comes from the viscous component (resistor, Fig. 1) introduced in the coupling between "extensive" and "intensive" variables, characterizing the first order differential representation of the process. All of these information are well known, but it is much less known that this parametrization hides some hyperbolic distances issues, to the singular boundaries (zero and infinity) according to $\frac{u}{v} = n$, an/or, $\frac{u}{n} = 1/n$, ratios which play a central role in the generalization of the above generic deterministic model that we would like to introduce and that we call Expon**a**ntial with “a”[1].

As we had suggested in previous scientific works initiated in electrochemistry [33], the physical treatment of Complex Systems, – characterized by a fuzzy interface of exchanges and inter-correlations between experimentalist (as subject) and object of experience (as devices) –, requires a very different approach than the classical differential equations controlled by boundary conditions. We were interested to the non integer differential equations, relevant for the simplest expression when the complexity is characterized by self-similarity. The theoretical approach of Complex Systems requires methods far from the traditional way of thinking and a long experience of these issues suggests a generalization of the Exponential concept named herein Expon**a**ntial (with “a”) able to be associated to the well-known universal functions labeled as Riemann zeta function (or more generally Dirichlet or L functions). To implement this aim we need to go through a trick suggested by the engineering of energy storage during the industrial development of Lithium Batteries [34]. Let us consider[2] a Fourier transform of a first order differential equation represented in the above complex plane and let us decide to share

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[1] Let us observe that under a real or complex expression the exponential function plays an absolutely central role in physics. Modified it, may cause a large number of consequences and among these, epistemological issues difficult to be estimated a priori.

[2] The Model of transfer of energy upon fractal interface (TEISI model) is based on a convolution between a geometrical dynamics associated to the self similar interface $L_d(t)$ with a fractal dimension d, and the local dynamical process of exchange $\varphi(t)$ of extensities (ion, electron, photon, momentum, spin etc) which can be of several natures (First order, diffusive, – Brownian, etc). The geometry then appears as the basis of the transfer function and the discretization requires a treatment in the Fourier space. The space-time relationship is in accordance with Mandelbrot relation linking the Hausdoff content η^d and the number of the parts N in the partition: $N\eta^d \sim 1$ namely for the TEISI model, $(i\omega\tau) \times \eta^d(\omega) \sim 1$. For $d = 1$ this relation gives meaning to the concept of velocity; for $d = 2$ this relation gives meaning to the concept of diffusion constant $(L^2 t^{-1})$, both characteristics attributing the status of Noether Invariant to the concept of energy. Therefore, if $1 < d < 2$, the Cole and Cole transfer function must be associated to the fact that $\varphi(t)$ is a first order transfer function, namely an exponential relaxation. The physical constant is no longer in adequacy with the Noether Principle and for this reason the patterning appears as incomplete with respect to the mechanical principle. As it has been shown since the design of TEISI model, this opinion forgets the implicit role of the phase angle in the analytic expression using fractional derivatives, and therefore to forget the data, contained in this expression, which allow us to settle easily the incompleteness.

geometrically the semi-circle into two parts, by referring the zero value of n on the pending extremity of the upper part of the diagram. This sharing is an adequate representation of the Fourier transformation of universal dynamics labeled as Cole and Cole dynamics, namely $Z = 1/[1+(i\omega\tau)^{\alpha}]$ observed in many physical Complex Systems [Figure 1] when controlled by power law dynamics [35, 36, 37, 38, 39] for instance $\varphi(t)+\delta_t \sim (t/\tau)^{1-\alpha}$. This dynamics involves rightful controversies about $\varphi(t)$ – convergence and Noether's constants, but the dynamics of Complex Systems takes a very simple and experimentally relevant form directly related to self-similar properties of the geometry in which the dynamics is embedded. These observations not only cannot be swept out without caution but can still be used heuristically to deepen our complex models [40, 41, 42, 43, 44]. Indeed in the simplest case the factor $\alpha = 1/d$ is related to the metric of the hyperbolic geometry of embedding [45, 46, 47, 48] (in general $3 \le d+1 \le 2$ is the fractal dimension of the $3D$ interface of exchange [49]). This observation highlights the key factors that are only scaling extensions of mere local properties averaged by the Fourier transform of the geometry filtered by the dynamics and, for example: $n \sim (\omega\tau)^{\alpha}$. But this observation imposes the underlying presence of a couple of system of vectors reference in $\mathbb{C}$ both being in competition for representing the phenomena. In the framework of the understanding of this competition a key point, probably the most important for our purposes is the following: the notion of hyperbolic distance to the edge $\frac{u}{\upsilon} = n,$ with $n \in \mathbb{N}$ is not only geometrically relevant for the upper arc of semi-circle but also geometrically true for the lower complementary arc. This feature reveals that the fundamental renormalization group properties of the embedding (causal process) includes, through an unknown functional equation at this step, the coupling of both arcs, through the couple of vectors references which is expressed via the phase angle (Figure 1). At this stage of the reasoning we know nothing about the coupling except that it is clearly an additional factor associated to a non causal effect associated to the experimentation. To illustrate the problematic let us consider a classical diffusion process [50].

$$\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2} = \left(\frac{d^{1/2}}{dt^{1/2}} - \sqrt{D}\,\frac{d}{dx}\right)\left(\frac{d^{1/2}}{dt^{1/2}} + \sqrt{D}\,\frac{d}{dx}\right) = F \times B$$

by using the fractional semi differential the diffusion equation can be split within a product $F \times B$ using fractional derivatives. The first, F is a causal relaxation process, but the second process B is a non causal factor due to a backward coupling (paradoxical time inversion). Therefore even using the classical reversible time some well known physical process can give birth to models taking into account physical externalities. These ones are herein controlled by the mathematical constraint imposed by the Pythagorean relation $(x^2 - y^2) = (x - y)(x + y)$. Fundamental question is, in the general case of the expon**a**ntial, the following: what is the nature of the function obviously derived from the exponential function that provides the link between the causal arc and the non-causal arc? What are the root and the mathematical expression of the functional relationship that characterizes this relationship? We shall show that the function is nothing less that the zeta Riemann function. We shall also show that the relationship is the well-known functional-relation characterizing zeta functions with respect to s. Thus the non-causal arc will emerge as a degree of freedom of our ability of mental representation of our environment when the time is a complex variable. In this framework the concept of "the intermediation for exchange, namely the project which gives its existence to the experimental object (the second arc)" will appear precisely like an archetype of the claiming of the freedom of the experimentalist when facing the ambiguity of any reference to be taken into account for treating a Complex Systems. This difficulty, which is put aside for instance in the usual engineering (for instance the role of the infinity in the treatment of diffusive processes or Fokker-Planck), appears here in full light, like it also appears through the concept of complementarity in Quantum Mechanics.

We name expon**a**ntial with "a" an exponential, divided into a couple of arcs as shown in Figure 1. The "a" points out the incompleteness of the unique causal component (upper arc associated to a Cole and Cole dynamics). We can highlight among other questions, the absence of inverse Fourier transform, and the absence of convergence of the power law relaxation. For these reasons the energy (Utility) disappears as invariant factor and, in this case, Emmy Noether's assumptions giving birth to the concepts of the constants of the motion have to be revised. The incompleteness observed, requires the addition a teleonomic term (the second arc), an addition which introduces the irreversibility of a time herein considered in $\mathbb{C}$; this irreversibility is related to the hyperbolic metric (curvature) of the manifold supporting the dynamics (leading the emergence of an entropic factor). To understand the rest of reasoning, we have to return to the links existing between the infinite set of the integer numbers that must be used as discretization parameters $\mathbb{N}$ and the zeta function Riemann.

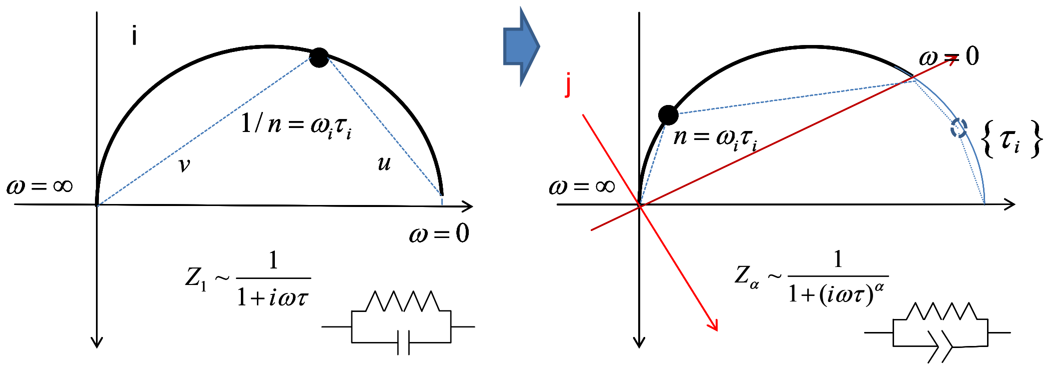

Figure 1. Illustration of the processing of the exponential function depicted in the Fourier space into an exponantial as defined in the text. The point characterized by the zero frequency (pointing out usually equilibrium) is herein not any more on the real axis but suspended in the complex plane. The incompleteness appears. Let us observe the fundamental role played by the referential of real axis in both within the mathematical expression but also in the epistemological status of this new kind of relaxation.

Let us observe that Laplace transformation of any dynamics characterized by a power law $\varphi(t) \sim t^\nu / \Gamma(1+\nu)$ can be defined by $Z(p) = 1/p^{1+\nu}$ if and only if $\Re(\nu) > 0$. Due to the fact that $1 + \Re(\nu) = \alpha \in \left[\frac{1}{2}, 1\right]$ this constraint is not fulfilled herein even if the current term of the dynamics is given by $n^{-\alpha}$ [34,51,52]. Accordingly even observed experimentally through spectral analysis, the inverse Fourier of the exponantial dynamics (or inverse Laplace transform), namely its temporal expression, cannot be found. Nevertheless the dynamic may be "extended" or "forced" in the complex plane, without any mathematical objection, according to n^{-s} with $s = \alpha + j\theta \in \mathbb{C}$. This action will be called "fibration" for reasons that will be given later. Under this mathematical form, the generic term of the extended dynamics, is nothing less than the generic term of the Riemann Series which defines the zeta function $\zeta(s) \sum_{n \in N} n^{-s}$. Thus there is a straightforward relationship between the zeta Riemann function $\zeta(s)$, exponantial dynamics and scaling properties. Likewise, the relationship between Riemann zeta function and the introduction of self-similar properties of curved geometries via the category theory [53, 54, 55]. The hyperbolic distance integral definition measured on the dynamics founds the value of n, and therefore $\zeta(s)$ may be defined geometrically upon the causal arc. We can show that this conclusion stays operative for the non-causal arc associated with $\zeta(1-s)$. Moreover the non-causal extension of the arc is related to the causal arc, in a very similar relation to the linking between a Fourier transform of power law and its temporal dynamics. Therefore, the correlations between ω and τ within $n=\omega\tau$ using the multi-decomposition in the prime-set, may base the linking between τ and $\zeta(1-s)$ realm and n and $\zeta(s)$ realm. Obviously this linking finds its roots within the functional relationship: $\zeta(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ and $\Gamma(s)\Gamma(1-s)=\pi/\sin(s\pi)$ relations which become the operators for depicting in the case of exponantial dynamics, something like a Fourier transform (averaging, distributions data, combinatorial organization etc), both arcs giving birth to an involution. In this framework the lower arc of circle (non-causal) is nothing less than the set of $\{\tau\}$ obtained by scanning the whole of experimental possibilities offered by dividing within different manners n into two integer parts. We shall prove later that the lower arc of circle, base of $\zeta(1-s)$ is a Kan extension of the upper causal arc basis of $\zeta(s)$.

To refer to the decomposition of diffusion process split between forward arc (causal arc) and backward arc non causal arc), let us observe that the factor required to define the expon**a**ntial concept – so often observed in physics [35, 37, 38, 39, 56, 57, 58, 59] – may be based on an expansion of the concept of capacitor namely $I \sim C\frac{dU}{\mathrm{d}t}$ onto a concept of "fractance" [34, 58] that behaves according to the following non integer differential equation $I \sim F\frac{d^{\alpha}U}{\mathrm{d}t^{\alpha}}$ where I is the flow of intensities, U is an extensity, C and F physical constants. The Fourier transform of the fractance behavior is hence given by $(in)^{\alpha}=(i\omega\tau)^{\alpha}$ involving non integer derivative [50]. From this point of analysis, we must be able to distinguished the operator required for the definition of the Riemann zeta function with $s=\alpha+j\theta$ from to the expon**a**ntial definition of a complex exponent, namely $i^{\alpha}\times(1/\omega\tau)^{s}$. The coherence must absolutely be searched. This compelling imperative opens the issue of the referential taken as the basis for computing the "fibration" of the real dynamics (via a rotation) $\theta(\mathrm{mod}\,2\pi)$. The coherence is obtained if and only if $j=-i\alpha$. This mathematical relation is an imperative for two reasons: (i) for understanding the issue related to the choice of the referential [60, 61] for describing the experimental reality, (ii) for taking into account the remnant term represented by the phase factor when we shall have to develop the "vibration" based either via θ or through t and (iii) obviously to give a physical meaning to the complex fractional derivation [62]. Figure 2 may be used for understanding this subtle but major point of the reasoning. Let us now introduce the categorical foundations of the above physical concepts [59].

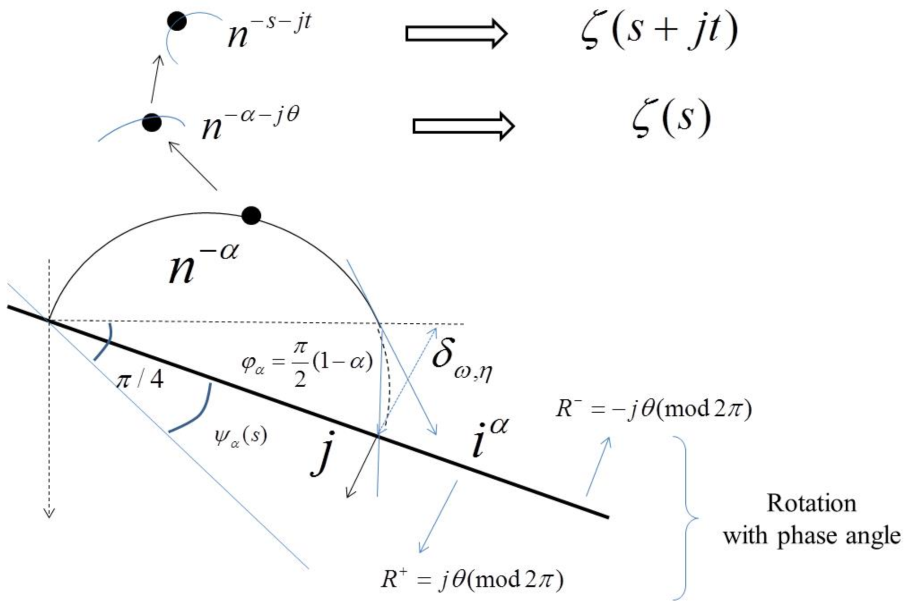

Figure 2. The exponantial basis of the zeta function construction, by "θ-fibration" (herein the initial step if $s = \alpha + j\theta$ *if* $-i = j^{\alpha}$). This picture, which is based on the non integer differential operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}} - k\frac{\partial}{\partial_x}$, highlights (i) the irreducible role of the phase angle φ_α in the implementation of the θ-fibration, (ii) the role of the sign of the θ rotation (imposed by the order of axes in the complex plane) on the appearance of a geometrical time irreversibility, (iii) the specific role of the axis characterized by $\varphi_{1/2} = -\pi/4$ authorizing a symmetric representation of the complex plane; an axis able to lock the geometrical time reversibility and to reduce exponantial dynamics to a mere diffusion process.

In accordance with the definition of the Riemann zeta function as series of distributions, it is possible to build step by step this function by starting from data based on exponantial dynamics such as explained here above via an operation named "fibrations" in reference to the fiber-categories [59].

III. The set of integers, *N*(*s*) – Memory and zeta function

As shown in some previous works, the Riemann hypothesis analyzed within the framework of physical issues sheds light on the universality of complex systems characteristics. The basic idea of the current work is to distinguish purely analytical aspects as expressed by the famous Riemann Hypothesis, and structural and geometrical aspects clearly associated to the physical approaches. The relationship between zeta function and a fractional dynamics able to be associated with fractal structure highlights the relationship between $\zeta(s)$ and self similarity. The duality of the two aspects (analytical and structural) is central to the mathematical approach that the authors claim to implement using categorical approach. Due to underlying fractal properties the authors have to relate the theory of category, recursive constructions, and self-referred algebraic structures to ensure the mathematical foundations of the role of zeta function [53, 54, 55]. The zeta Riemann function $\zeta(s)$ given by the following mathematical expressions is the junction between the different points of view. The first equation emphasizes the characteristics of a sum of integers and the second equation underlines the characteristics of a product involving prime numbers.

$$\zeta(s) = \sum_{n \in N} n^{-s} = \prod_{p \in \wp} (1 - p^{-s})^{-1} \tag{1}$$

with $s = \alpha + i\theta$. In this equations $n = \prod_{p \in \wp} p_i^{r_i}$ the integer may be divided within primes p_i as usual. The equality points out the crisscrossing between multiplication (inversion written according to parallel diagram) and addition (written according to a series diagram). This duality reminds the properties of the duality between exponential function and the logarithm function and suggests that the zeta function might be like a generalization of an exponential operator, point of view being able to base the heuristic suggestions given by the above physical arguing (chapter 2). We thus must base categorically this duality for understanding how the self similarity is arithmetically hidden behind the zeta function and therefore, according to Voronin and Bagchi [26], why the Riemann's hypothesis is finally true. The first equality refers clearly [28, 29, 30, 31, 8] to the additive monoid $(\mathbb{N} + \leq)$, while the second refers to the multiplicative monoid $(\mathbb{N} \times \geq)$. Let us note that in $\zeta(s)$ the entanglement between the couple of ordering (series and parallel) is for instance illustrated by the following development: if we write $q = p^{-s}$, then $(1 - q)^{-1} \sim 1 + q + q^2 + ... + q^n + ...$ relation highlighting a succession different of the natural one. But the operator of product Π reports that each term is then built according to a parallel procedure that authorizes freedom with respect to the labeling of the multiplicative order.

As we reported above we can wrote the dynamics in the infinite discrete set: $\{n = \omega\tau\}$ with $n \in \mathbb{N}$. By doing so we build a 2*D* a vectorial space of set $\{n\}$ punctuated by singular states n depicting step after step the overall set $\mathbb{N}$. The Riemann zeta function defined by equation 1 is nothing less than (Figure 3, sticker) the mathematical trace of the exponential operator upon this basic space when after assigning the factor s to the scaling, n is represented in the topological discrete infinite space whose referential is built upon the axes given by $\log(p_i)$ if $p_i \in \wp$ the set of primes. Thus the zeta function associates implicitly the set $\mathbb{N}$ and s together and therefore we'll call $N(s)$ a set simply a translated through s as a copy of the set $\mathbb{N}$ (Figure 3). This translation carries the self-similarity of $\mathbb{N}$, because that it is simply an external change of scale of the mesh of the $\mathbb{N}$ without any modification of the small categories $\{n\}$. This may be extended by introducing an additional factor according to the transformation: $s \to s + jt$ (Figure 3)[3].

Figure 3 gives a clear idea of how and why zeta function is mathematically associated to the set of integers $\mathbb{N}$ and, due to the decomposition within suitable normalized space, is virtually anchored to scaling structural characteristics of the set $\mathbb{N}$. So for going deeper in the issues involved by the zeta function and the Riemann's hypothesis, a small category must be considered, namely the set of integers.

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[3] Nevertheless recall that this scaling law is expressed in terms of the complex variable $s = \alpha + j\theta$, with $j^2 = -1$; is linked with the exponantial expression only through the phase part α of the factor s, and note that, depending on the referential chosen, the complex factors (of the physical process) and j above are related together through the following rotation of the complex basis $-i^{\alpha} = j$.

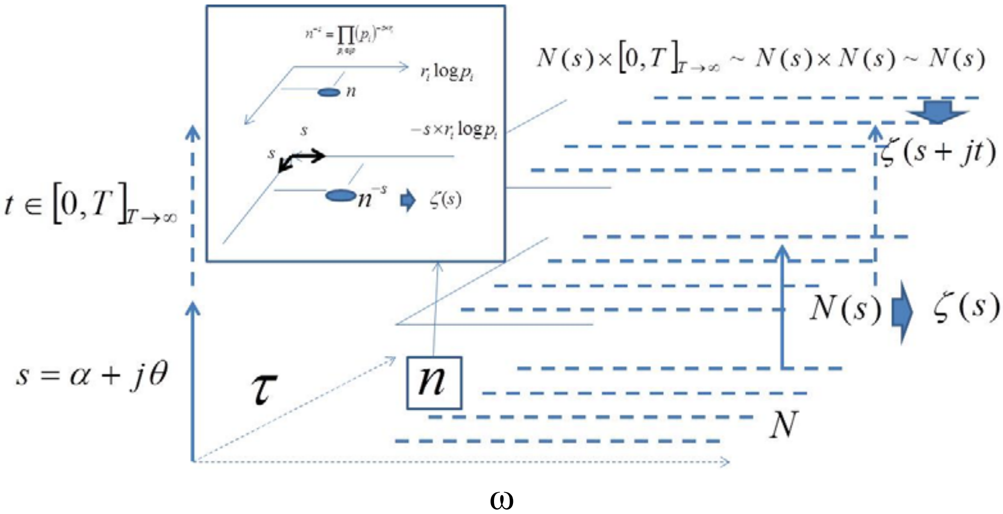

Figure 3. Analysis of the extension of the set of integer (herein share into 2 spaces ω and τ) via the extension of the scaling factor "s". In the sticker, we depict the construction of the infinite, discrete and countable topological space of the natural numbers ℕ based on the decomposition of every integer into primes. Each axis is given by the logarithm of a prime number. Every natural number has a unique position in this space. Due to the function of the logarithm in the representation of the space, this one responds linearly to any s − scaling factor. The space becomes $N(s)$ and zeta function is nothing less than the trace of the exponential operator upon this space.

Therefore question of the filling and of the draining of such a set is open. Indeed at each place n we can write a piece of information depending on n. Thus, $N(s)$ must be considered like a memory (Figure 4). It can be seen either like an initial categorical object (at disposal to be filled) or like final object (after draining of all information put inside). In both cases, zeta function stays *a standard reference* on ℕ. For our purpose it is relevant to consider the filling of the memory with an analytic function written as a dual expression of zeta function, namely: $g(s)\sum_{n \in N} a_n s^n$. To facilitate the reasoning we can define another analytic function $f(s) : \log[f(s)] = g(s)$ relevant to take into account the scaling properties. The duality $n^{-s} \leftrightarrow s^n$ commutes explicitly in the convolution the operator of addition and of multiplication while the exponentiation keeps the duality: $(n_1 \times n_2)^{-s} \leftrightarrow s^{n1+n2}$. Thus we have at disposal a possible comparison between couples of traces: (i) the reference $\zeta(s)$ and (ii) a memorized analytic function $f(s)$. The mathematical content of the structure of the memory and the duality imposed to the contents, authorize a comprehensive understanding of Riemann's Hypothesis, not only within its analytical expression (obtained as a by-product of the structural arguing) but as the result of an asymptotic comparison between the initial empty state of the memory and a couple of empty states obtained after having chosen how to drain the full memory (Figure 4). This issue requires previously managing how to expand the complex part θ of the factor $s = \alpha + j\theta$ to implement the transformation $s \to s + jt$ (figure 3). As we show schematically in the Figure 4, the existence of two distinct methods for draining the fulfilled memory led to the examination in depth of the issue of a questionable identity between the initial state of the memory and its final ambiguous state. This question introduces categorical subtleties that the alone mathematical analysis cannot merely revisit leaving aside structural properties of the limits and co-limits, hence the current difficulties observed for solving the Riemann conjecture which is clearly an issue of arithmetic entanglement involving infinity.

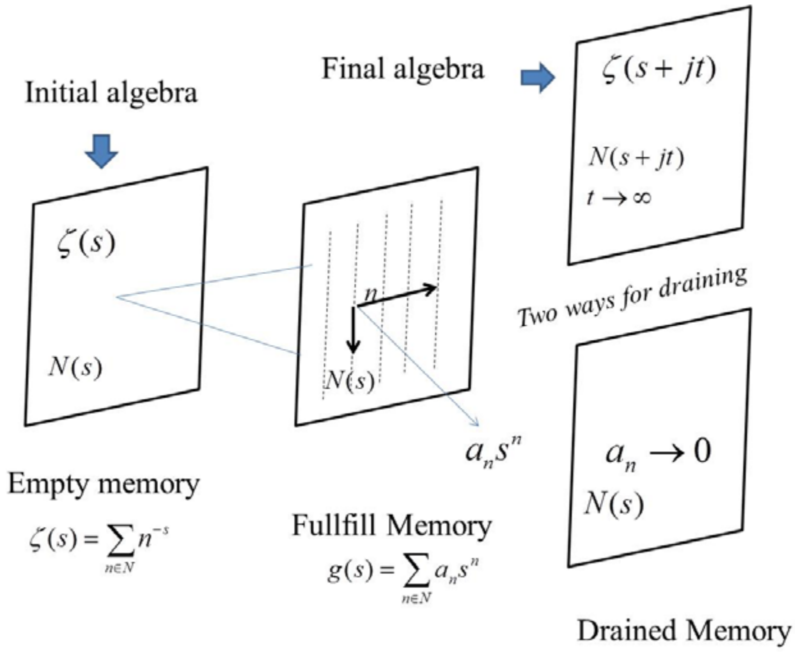

Figure 4. Categorical point of view depicting the relationships between, (i) the state of an empty $N(s)$ memory (whose exponential trace is given by the zeta function), (ii) the $N(s)$ memory fulfilled by any analytic discrete distribution $g(s) = \sum_{n \in N} a_n s^n$ and the couple of methods able to be implemented for draining this memory. The result is then an empty memory. One of the method for draining leads to the proof of Voronin's universal theorem.

That why the understanding requires to go deeper within the category theory concerning the properties at the edges. We'll see why the consistency of the arithmetical objects of this theory requires the taking into account of the internal closure of the addition on the multiplication and we shall prove that this closure may be expressed through a categorical concept of adjunction applied on a pair of reciprocal functors. It will be shown how this addition opens onto the emergence of self-similar structures, situation that explains the universality of such a structure and will justify the validity of Riemann Hypothesis.

We also understand at this stage, how the physical approaches – that analyze local equilibria between exchanged extensities (adjunctions) in complex structures characterized by self-similar properties (fractals) – may involve the Riemann zeta function seen as Universal Reference (dynamical categories) and why these data open new perspectives on physical approach of the Riemann Hypothesis [63, 64].

IV. Why Riemann's Hypothesis is true: categorical approach

A. Arithmetic order, closure and Self Similarity

By identifying each of them by a characteristic number n, the set of natural numbers $\mathbb{N}$ may be used to compile lists of objects in any type of sets; this property allows us to store information (data or software) in a rational way. Nevertheless there are many ways to establish such lists, methods more or less structured depending on the purpose. Let us recall that $\mathbb{N}$ can be seen as categorical object, an even more precisely, as initial object with respect to the recursion operation. This property implicitly intervenes decisively in the reasoning set out in this note. The aim is to browse recursively and systematically, the topological countable set of objects, operation that always requires the use of an initial categorically object, namely $\mathbb{N}$. From a mathematical point of view, the notion of order is then involved, therefore similarly the categorical concept of functor. Indeed, any order structure defined by an inequality, $\leq$ and associated algebras may be related to the presence of arrows $\rightarrow$ associated algebra, and then the relationship between G categories and its opposite $G \rightarrow G^{op}$

arises and as well the concept of adjunction. Therefore $\zeta(s)$ may be structurally attached to the notion of order [65]. A set characterized with an order relationship inside is named Poset. More precisely:

- overall linear order on $\mathbb{N}$ may be associated with the additive monoid $(N,+,\leq)$ and the definition of a chain which may be associated to physical models of series of process. Every integer must imperiously be lower or higher than another. This monoids bases the definition of additive zeta series.
- partial order may be associated to the multiplicative monoid $(N,\times,\geq)$ asserting that $r_1 \leq r_2$ if and only if r_1 divides r_2. In $\mathbb{N}$, for instance this equation leads the definition of the set of primes. This monoids bases the definition of the Euler definition of zeta function.

The second order relation is of a different nature of the first one; two integers are not always comparable and there is an infinite amount of countable integers that are comparable with only themselves, namely primes. The primes form what is called anti-chain for this peculiar order relation. With this second order relation the set $\mathbb{N}$ is an only partially ordered set. Its structure is actually richer and constitutes what is called a lattice: for any pair of elements x and y there is only an upper bound, namely the Lower Common Multiple (*LCM*), and a lower bound which is Greater Common Divisor (*GCD*). Both give birth to associative algebras. The second order relationship will be useful to give better structuration of artificial memories, thanks to partitions. Obviously the first ordering rule also provides a lattice structures for which the *max* and *min* operators replace the *LCM* and *GCD*. This general point of view finds its basis in Janos Aczel's works about the Associative Algebra [66, 67, 68]. An example of algebraic rule that emerges from the Associative Algebra is given by the relationship between Lower Common Multiple and Greater Common Divisor, namely $LogLCM(p,q) = Log(p) + Log(q) - LogGCD(p,q)$. If $GDC = 1$ both figures are primes. Let us observe the entanglement of addition and multiplication in the equation. Indeed, J. Aczel highlights the requirement to match together the couple of chains and the couple of associated algebras to reach an overall coherence of a final associative algebraic structure. In the peculiar case of a parametrization through a real parameter and of a continuous function, algebra gives birth to dual functions named Logarithm and Exponential. We would like to show that in the case of discrete order distribution, the duality is then based (i) on analytic functions (complex dynamics) and (ii) on the relevance of the zeta functions (complex universal reference) taken into account as a reference (trace) in $\mathbb{N}$. The coherence is expressed through self-similar characteristics of the emerging geometry associated to the group of symmetry thus explicitly created (Figure 5). This issue of coherence is obviously central within advanced developments of our arguing about zeta Riemann function. The understanding of $\zeta(s)$ structural properties is based upon a couple of antinomy choices: (i) an initial clear distinction between addition and multiplication and (i) a final will to close the coherence of a relationship by matching both dual aspects within adjunction operators. This approach requires two additional comments

- traditional factorization based upon the prime numbers minimizes the number of factors required to achieve it. This fact suggests the presence of an underlying Galois connection used in abstract interpretation and programming languages. Let us recall that Galois connection is nothing but a pair of adjoint functors between two categories that arises from partially ordered set. In this context the upper adjoint may be associated to the right adjoint while the lower adjoint may be associated the left adjoint in rather general situation (no transformation of Posets into categories in a dual fashion).
- The construction of the field $\mathbb{Q}$ from the semi ring $\mathbb{N}$ (or $\mathbb{Z}$) is performed through the Dedekind's Cutting. This method is asserted for any kind of order structure giving birth to the Dedekind-MacNeille completion theorem and $\mathbb{Q}$ is then characterized by a order structure which is nothing less than an extension of the $\mathbb{N}$ (or $\mathbb{Z}$) order with $\frac{a}{b} \le \frac{a'}{b'} \leftrightarrow ab' \le a'b$ if $a, b \in \mathbf{N}$ (or $\mathbb{Z}$). Let us observe that the order relation $\le$ requires mainly additive operations. Practically $(N, \le) \subset (Q, \le)$ is a strict inclusion because $\mathbb{Q}$ is dense when it is not the case for $\mathbb{N}$. Nevertheless we can create equivalence by choosing another order relation for $\mathbb{Q}$. This equivalence is obtained by choosing a multiplicative rule of order instead of additive one. The reaching of the equivalence leads the emergence of self similar structure and scaling invariance is led by the involvement of categorical epimorphism. This equivalence carries *in*

nucleo the concept of fractality. This emergence can be reached by observing that if $q = \frac{m}{n}$ this ratio may be written $q(m,n)$ and may be associated with the sum $(m + n)$ representing a path in $\mathbb{N}$ space whereas that it may also be represented as a state in $\mathbb{N} \times \mathbb{N}$ space. The question of the equivalence $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ is therefore again open.

Due to the similarity with the self-similarity [69], is obvious that the complexity of the proof of analytic the Riemann Hypothesis must be associated with the entanglement of product and co product (addition) hidden behind a self-similar structure of $\zeta(s)$. The basis of our reasoning is to turn the analytic expression given by Riemann into a structural issue able to open the way for unraveling of the analytical entanglement. This reasoning finds its origin in the fact that identical complexity is existent into the categorical structures (with the entanglement of coproduct and product), if we consider in this framework the ordered structures, the associated categories and algebras. The factorization and the inversion of ordered morphisms give then birth to lattices with upper and lower bounds and the introduction of infinity creates obviously some problems. In the framework of associative algebra, the taken into account of the orientation of the morphism using monoids – that intrinsically excludes the inversion, may mimic the analytic questions, and therefore offers within category theory, an indirect approach of the Riemann Hypothesis. Two monoids must then be considered $(N+)$ and $(N\times)$, their entanglement being expressed by an asymptotic closure of a dual structure (Figure 5), gives a diagrammatic approach of the problematic. This diagram, -that points out the fundamental role of the orientation of the arrows-, puts the emphasis implicitly upon the role that must be played by the factorization (inversion of the arrow carried by the multiplicative monoid $(N\times)$).

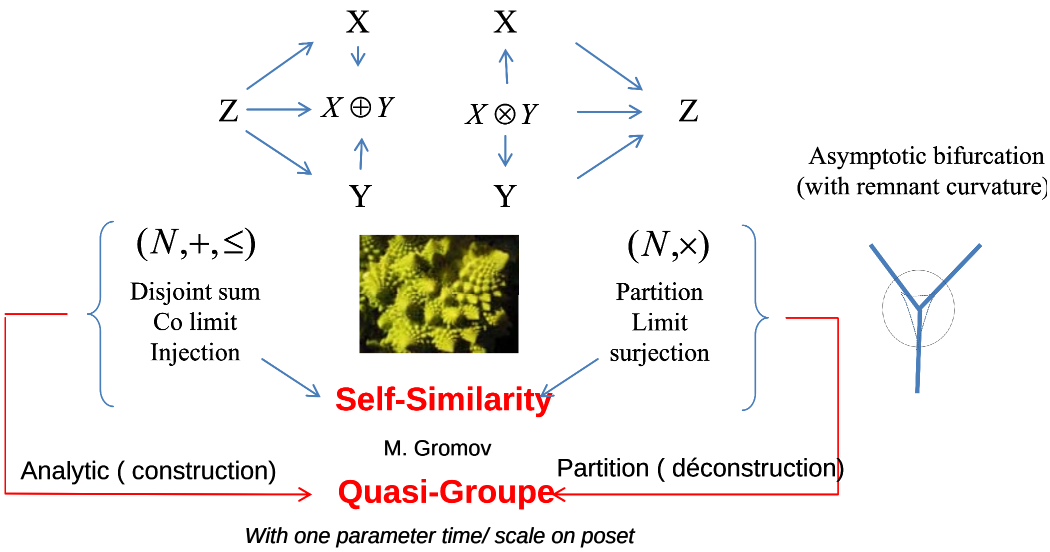

Figure 5. Gives the analysis of the association of multiplicative and additive monoids with emergence of the self-similarity. It can be shown that the Riemann zeta function is the natural trace of the exponential operator upon $\mathbb{N}$ and therefore zeta being characterized by self-similar properties (according to Bagchi inequality), the Riemann hypothesis is true.

The closure of the problematic is not obvious and requires mathematical guaranties of the relevance of the operations then imposed. Practically a rightful closure leads a couple of constraints: (i) to conceive the role of a functorial monad and (ii) to lift the monadic morphisms within a specific category named Kliesli category for taking into account the relations between the set of objects on which the morphisms are defined. This taken into account leads also naturally to the arising of self-similarity and fractal structures. But as it has been pointed out elsewhere [8, 59] zeta function is naturally associated with such properties. Moreover, this function puts at disposal its functional equation, equation that through its symmetries looks like by some aspects at an extension of Fourier transform. Therefore the key component of our reasoning must be focused on the question of ability to inverse the functors (pulling back the arrows). That is why the question of categorical adjunction is the next step of the arguing.

B. Categorical adjunction and self-similar structures

As seen above there are, at least experimentally, a link between the zeta function, the power law dynamics in physics and the self-similar geometrical structures that shape such dynamics [22, 21]. Aiming to give a categorical perspective to the issues involved by these observations we can show that it is possible to categorically address the issue of self similarity. It appears that this theory (category) is particularly well suited to grasp the scope of morphisms in scale conveyed by the self-similarity. Indeed self-similar structures can be understood as objects characterized as fix-points (attractor and anchor) of functors in a certain category. In very fundamental framework of categorical thinking, the theory of arrows, such a fixed point can be designed as a backdrop erected on the base of a couple involving two opposite functors.

$$\lambda : A \to B; \rho : B \to A \tag{2}$$
$$\forall p \in A; \forall q \in B : p \leq \rho(q) \Leftrightarrow \lambda(p) \leq q$$

where λ and ρ are named adjoint functors. It is claimed that this adjunction, giving birth to a functor such as $T = \lambda \circ \rho$, is also the core of the operations of the theory of Poset. The preservation of the order is the main reason of the prominent position of the adjunction. Indeed if we name Ideal Principal (IP) the set of the smallest objects, and Filter Principal (FP) the set of greatest objects (refereed by the order of the Poset) it can be ensured that each functor has at its disposal one adjoint to the right (respectively to the left) if the inverse image of IP are also an IP (respectively if the image of FP are FP) [70]. The first operation (residue) preserves the Union of sets (joints) whereas the second (residual) preserves the intersection of sets (meeting) and reciprocally. A mere mathematical development shows that adjunction are strongly linked with the concept of closing (illustrated by fractal boundary) and of kernel (illustrated by the attracting point for instance in Newton method of approximation). These notions possess their equivalents in topology and thus there is a morphism between a topology and a scheme of adjunction. In this framework the concept of scaling imply naturally the issue of the concept of vicinity in scales order. Indeed the attractor is given through the application *TX→X*. The main issue concerns the question of rightful categories and the rightful functor able to explain the universality observed when it is anchored upon the self-similarity. The answer is the concept of Monad lifted in a relevant category named Kleisli category (category of relations (Figure 6))[4].

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[4] Indeed the reasoning starts from the concept of monoid. If we consider a binary operation $C \times C \to C : (x, y) \to xy$ structuring any languages using words and extended to $(x_1,..., x_n) \to x_1...x_n$, we can write $T(C)$ the set of series and we can give a monoidal structure to M through the application $\xi : T(C) \to C$, namely $(x_1,..., x_n) \to \xi\ (x_1...x_n) \to x_1...x_n$. Therefore ξ axioms is not but that the algebraic composition rules according to Fig. 6 and $\varepsilon_C : x \to \{x\}$ and $\mu_C : TT(C) \to T(C)$ namely, $((a_1,..., a_n)...(a'_1,..., a'_{n'}) \to (a_1,..., a_n,..., a'_1,..., a'_{n'})$ giving naturally the associative rule. Thus the monoid is "categorized" and gives birth to the concept of Monad in the category O. Practically this Monad is a triplet of a functor $T : O \to O$ and a couple of natural transforms $\varepsilon : 1_O \to T, 1_O : T \to 1_O : T \xrightarrow{\varepsilon \times 1_T} T \circ T \xleftarrow{1 \times \varepsilon} T$ and $\mu : T \circ T \to T$ An algebra is naturally associated onto the Monad according to the ~~the~~ pair $(C; \xi)$ following rules: $C \in \mathcal{O}$ and $\xi : T(C) \to C$ is a morphism of O validating the diagram given in Figure 6, base of the concept of monoid. If (D, ζ) is another algebra a morphism between algebras exists in O $(C; \xi) \to (D; \zeta)$ namely $f : C \to D$ and $T(f) : T(C) \to T(D)$ such as $\xi : T(C) \to C$ and $\zeta : T(D) \to D$. The set of the algebras that can be built with O and T is itself a category $\mathcal{O}^{\mathcal{T}}$ named *Eilenberg-Moore Category*. There is a forgiving functor $U: \mathcal{O}^{\mathcal{T}} \to O$ namely $(C; \xi) \to C$. U throw back the isomorphisms. U has an adjunct F at left $F : O \to O^T$ namely $C \to [T(c), \mu_C]$ the unit of the adjunction is given by $\varepsilon : 1_{\mathcal{O}} \to UF = T$ the counit is $\eta : FU \to 1_{\mathcal{O}^\tau}$ such as $\eta(C, \xi) = \xi$. The algebra $[T(C), \mu_C]$ is then a specific case named *Free Algebra.* The full sub categories of $\mathcal{O}^{\mathcal{T}}$ generated from free algebras is equivalent to an $\mathcal{O}_{\mathcal{T}}$ algebra named Kleisli category. The objects of $\mathcal{O}_{\mathcal{T}}$ are the O - objects and mainly the morphism $f : C \to D$ in $\mathcal{O}_{\mathcal{T}}$ is the morphism $f : C \to T(D)$ in $\mathcal{O}$. The composition of morphisms $f : A \to B$ and $g : B \to C$ may be understood as the composition in $\mathcal{O}$ such as $A \xrightarrow{f} T(B) \xrightarrow{T(g)} TT(C) \xrightarrow{\mu_C} T(c)$. Therefore the composition of morphisms ensures a perfect coherence with monadic structure. Therefore have at disposal the functor $\mathcal{O}_{\mathcal{T}} \to \mathcal{O}$ $C \to T(C)$ and $(f : C \to D) \to \mu_D \circ T(f)$ this functor reflected the isomorphisms and has an adjunct functor at left $\mathcal{O} \to \mathcal{O}_{\mathcal{T}}$, $T(C) \to C$ and $(f : C \to D) \to \varepsilon_D \circ f$. For summarizing, any adjunction gives birth to a monad and among all the adjunctions that rises the same Monad the Kleisli adjunction and Eilenberg-Moore are extreme, respectively as the smallest and as the greatest.

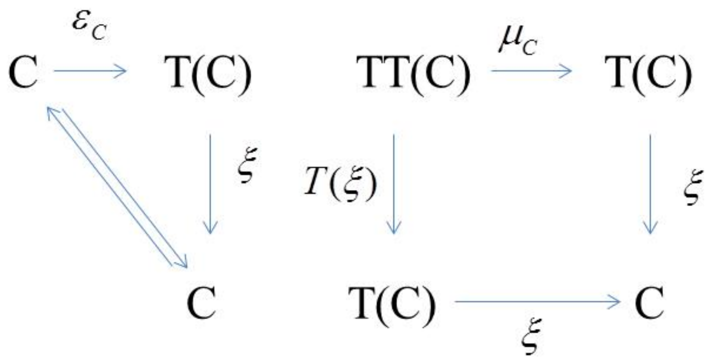

Figure 6. Conditions of the validity of the monadic algebras. This abstract expression is nevertheless merely based on the fact that any representation of our environment by the means of a language leads to an associative behavior $[(a_1...a_m)...(n_1...n_p)] = [a_1...a_m...n_1...n_p]$: characters, words, phrases, novels... libraries, A natural transformation like ε_c allows to transform a functor in a category into another functor in another isomorphic category. We understand herein why the theory of category founds the Computer Science as a science of representations.

In practice, the concept of adjunction, hence like in optic, the concept of reflection, leads to consider the notion of optimal approximation of an object analyzed through a certain functor (or an operator). This point of view on the adjunction that leads to the will of an optimal representation of a categorical object can be generalized to a set of objects. Certainly, by defining the object through performative operations (set of functors) we introduced some blur and the object must be then fuzzy, but its definition remains consistent with its handling and this operative point of view appears especially well suited to the aim defined by starting from the algebras associated to the operators. The important point is that this definition of a class of objects is unique and that a set of solutions always exists if the categories considered are categories called small categories, which means that any object of the category is itself a set. It is easily seen that the above considerations can be reversed since our relationship to the world is necessarily that of a functor (action ordered in the time). If the world always appears in a fog (the reality is veiled), so with some uncertainty, an ability to think of new objects and new morphisms between items is nevertheless given, as soon as the operator, (herein operator of creativity), allows to think algebraic coherence, topology and metric of the environment of the object. This posture becomes a teleological position of the operator. Nevertheless this position is neither anti-scientific nor anti-rational, if the mathematical concept used to implement the dream about a new realm is categorically relevant. The concept of Kan extension (Figure 7) gives precisely this possibility.

For what we are concerned herein, the notion of Kan extension: extension is firstly related to the idea of reversing a functor F, namely the implementation of the functor F^{-1}. Everyone understood that this inversion and no less legitimate than an experience justifies the optimality of a model that motivated this one. In practice this inversion is based arithmetically on the computation of fractions. Moreover the notion of Kan expansion generalizes the concept of adjunction by using an annex functor G as we will show. If the fibration is likened to a beam of light, Kan extension looks like an optical image. In practice this notion is associated to the Yoneda Lemma whose deeper meaning is related to an optical template, namely, a given object can be observed from a multiplicity of different "addresses" and if that is the case then a set of morphims exists between these different addresses [71].

At the fundamental level the concept of extension is based on the demonstration that there is a canonical implementation authorizing the building a new category from the existence of a given category qualified category "comma". We will not dwell herein on the proof of this lemma based on the combination of a pair of functors (F,G) and the associated construction of a cell of a square scheme using two functors that naturally emerges (U,V). An important special case of this type of construction is the introduction of the category "elements" of set (in connection with the concept of set exponentiation) denoted *ens*, thus $F\colon A \to ens$. We call Kan extension functor $F*:(B,\ ens)\to(A,\ ens)$ a functor that preserves the colimit. Yoneda's lemma (representable functors), ensures that there is a right adjoint functor to $F*$. We can show that the same properties are true for any co-complete category *ens* and in these conditions if we consider two functors $F : A\to B$ and $G : B\to O$ then it exists two functors K and H and a natural transformation $\alpha : G \Rightarrow K \circ F$ giving rise to the diagram given in Figure 7.

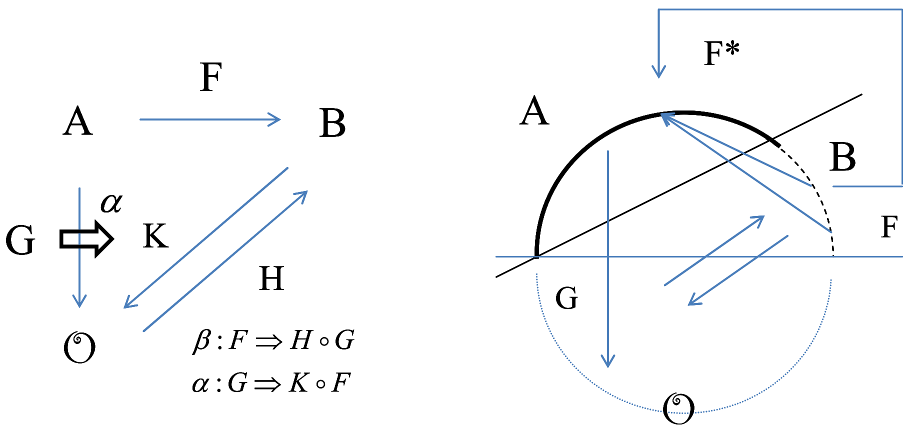

Figure 7. Schematic depiction of Kan extension, standard which manages the enlargement (the forcing) of categories and morphisms. This diagrammatic representation is herein associated to the concept of exponantial to show how these dynamics hold inside themselves, implicitly the opportunity of a natural Kan extension. $\alpha = 1/d$ for instance contains the information about θ fibration $\alpha \rightarrow s = \alpha + j\theta$ (complex extension) anticipating the Voronin one: $s \rightarrow s + jt$.

Thus the concept of Kan extensions (which in category theory implements at its level of abstraction something like the concept of Paul Cohen's forcing in the set theory) assures the ability to extend categorical objects along suitable functors, so use the self-reference to fulfill a possible incompleteness by using some annex morphisms.

Coming back to the adjunction, because we are concerned by a categorical fix-point (anchor or attractor of the functor) we can reduced the issue to a special case of Monad, called idempotent Monad. The category theory assures that a canonical procedure exists that can give birth to such an Idempotent Monad starting from any Standard Monad. In the interpretation which is that of this note, these observations mean that, by using a canonical operation, we can – from any set of phenomenological data (e.g. a physical experiments) – build a model that highlights self-similar properties [72]. Similarly, when in the presence of such Idempotent Monad, the categorical model corresponding to it can be reduced to a physical model. Saying categories means that the situation can be described in mathematical terms through classes of objects and classes of morphisms between objects. In this case, namely if self-similarity is observed, the objects that emerge are those that are the fix points for the Monad and then the morphisms that emerge, are coherent with regard to this class of objects. They are said to be orthogonal with respect to this class of object [73]. The morphisms depict of changes that are allowed on objects or physical actions that are possible on the same objects. The orthogonality relation expresses the matching between the class of objects considered and the class of transformations that can be applied. The self-similarity emerges precisely when these conditions are fulfilled. It can be proved that the ad equation is the source of the universality of a self-similarity very generally observed in complex systems. Such systems are systems in which the internal transformations and objects of these transformations emerge jointly. We must add to these observations the following categorical comments: the analysis of the adjunction operators and of Kan expansion, formally include the reversing functors (arrows reversal). But this inversion requires installing a computation of fractions. Algebraically it can be reduced to a transition from a set $\mathbb{Z}$ into the set $\mathbb{Q}$. This operation carries implicitly with it a notion of localization that can also be interpreted in topological terms (open set or closed set) and in terms of differential operators by using graduated algebraic structures. Therefore, addressing self-similar issues through the theory of the category naturally opens up useful perspectives in terms of new structures that can solved physical questions still in shadow. Among them the relationship between the Riemann Hypothesis and the paradigms of the Quantum Mechanics which consider as scientific objects only the objects able to live in an eternal cyclic time [17, 18, 56].

The core of the reasoning is then the following: The operations of adjunction and Kan extension are based onto somehow on the capability of reversing the functors (arrows). All previous approach therefore leads arithmetically at operate a calculation of fractions. It can be shown that the definition of Kleisli category can be exactly reduced to these operations; practically this method manages a fraction as a representable functors according to the diagram $\eta_Y : Y \to TY \leftarrow X : f$. Usually and according to the classical thinking of chains we are led to conceive is said fraction at left like: $\frac{f_n \times F_{n-1} \times ... \times f_1}{\eta_n \times \eta_{n-1} \times ... \times \eta_1}$. However we have seen that if every monad is induced by a pair of adjoint functors the result is not unique. Among the possible solutions two types of adjunctions are considered for characterizing the categories: the first called Kleisli is associated with an initial adjunction within the meaning of the source of the arrows, while the second called Eilenberg-Moore is associated with a final adjunction. But it can be shown that if the Monad source is idempotent, so then both categories merge their adjunctions and that the categories are isomorphic. Since it can be shown that any Monad can be a source of an idempotent Monad, the self-similarity associated with this characteristic is a universal feature of all processes based upon a Monadic structure, therefore upon adjunction. In physics it is particularly the case if, locally the process of construction and deconstruction of objects are present together within a local steady state, or in a localized loop. We would like to insist on the concept of localization which precisely emphasizes the reasoning upon the calculation of fractions, and therefore upon the issue of prime numbers and ideals. The localization is the formal inverse of the constructive method for algebraic structure. A localization of a category $\mathcal{O}$ associated to a collection of morphisms W is an universal methods through which all the morphisms W become isomorphism. The canonical and historical example is the transition from $\mathbb{Z}$ to $\mathbb{Z}(1/2)$ or the passing $\mathbb{Z}$ to $\mathbb{Q}$ with inversion of prime numbers $...f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + ...$ which opens on differential calculus, generalized for order structures and categories. An extensive analysis of above categorical issues can be found in the following references: [74, 75, 76, 78].

C. Fibration of the set of integer and exponantiation

Paragraph 3 shows very clearly that the definition of the zeta function is based upon a discrete topological space. This space is countably infinite and isomorph at $\mathbb{N}$. However the definition of zeta introduced a notion of scale $s = (\alpha + j\theta) \in \mathbb{C}$. Voronin's works and those of Bagchi argue that we must extend linearly this complex form according to $s \to s + jt$ namely to infinity, for understanding the relevance of the approximations of analytic functions. As, by product, according to Bagchi, we obtain the relevance of the Riemann Hypothesis. We would like to prove that this extension may be reduced to a categorical fibration that is needed to completely define the algebraic environment of zeta function. This "fibration" also appears as shown in Figure 2 if we study zeta function by starting from physical problems supported, as noted in the paragraph 3 by the concept of expon**a**ntiation (with "a"). However in this case the basis of the scaling properties is not any longer s but only α. We will see that this tiny difference, while driving similarly to the validation of Riemann's conjecture, broadens the perspective and opens the door to in-depth understanding of the incompleteness properties of Complex Systems. We will first consider the inequalities of Voronin and Bagchi in light of the theory of categories and then we will point the difference that mark the use of the concept expon**a**ntiation (with "a").

The Voronin theorem asserts the universality of the zeta function – based upon the exponentiation of the $\mathbb{N}$ set through n^{-s} series – for approaching any type of analytical function defined over the same $\mathbb{N}$ set – through analytic series based upon the dual function $a_n s^n$–. Let us note the duality of both expressions [24, 25]. This theorem may is expressed as follows: for any analytic function $f(s)$ and when the complex variable s belong to a compact set K with $\alpha \in [0,1]$ this function can be approximated by using an universal function (for example the Riemann zeta function $\zeta(s)$) accordingly to the following equation:

$$\forall \varepsilon > 0, \lim_{\inf} [\frac{1}{T} mes\{t \in [0,T]\} : \max |\zeta(s + jt) - f(s)| < \varepsilon] > 0. \quad (3)$$

Therefore, any analytic function (for instance associated with causal physical dynamics) can be compared with a zeta abstract function shifted in the complex set, accordingly with $s \to s + jt$. This approximation can be built without any consideration of the physical (analytical) processes. This function having few relationships with the concept of causality it is easy to understand that the Voronin theorem and the zeta Riemann function open together a broad epistemological perspectives about systems more complex than the purely causal ones. Nevertheless this formula involves the following paradox: why the inclusion of $s = \alpha + j(\theta + t)$ with $t \to \infty$ in $\mathbb{R}$ does not just stay in the $\mathbb{C}$ area but requires special $\mathbb{N}$ extension. Through the role of the angle θ given in paragraph 3, this mathematical requirement reflects the role of the singularity at the infinity, namely if s is defined in a compact class K, the "fibration" requires the definition of a "fibration-category". In other words, as shown by the physical model, t can not necessarily be reduced to a θ extension because if the minimum of t in its set of definition is 0 the minimum of θ may be different as shown in physical observations [22, 33, 34], the validity of the Riemann hypothesis being reach precisely if and only if the set of definition of θ and t fits together, in the frame of integer set *(t,θ/mod2π)*. Therefore this matching is not insured, and this constraint must be considered carefully.

First, let us start from the definition of $\zeta(s)$ and $f(s)$ by including the set of their ℕ-basis as given in the figures 3 and 4 respectively $N(s)$ and $N_f(s)$ with $s \in \mathbb{C} \subset K$ if K is a compact. ℕ is discrete and infinite. Let us observe, figure 3, that the definition of this class of fibrated-categories [31], seen as a layered structured (Figure 8), needs the presence of a transformation between addition into a Cartesian product in the ℕ-plane[5] Indeed the "fibration" may be expressed by a transition from ℕ to ℂ reduced to a moving of the basis which may be eventually be reversed by projection $s + jt \to s$ that operation suggest immediately that $-t$ the inverse of t must be considered as obvious in the above argumentation. It follows that within a ℕ geometrical perspective, a Cartesian product of the space $N(s)$ with an additional space required by the presence of the complex axis j parameterized by $t \in [0,T]$ or $t \in T_K$ emerges naturally. But according to Voronin theorem $T \to \infty$, therefore the domain $[0, T]$ has to match ℕ up to the infinity. Doing so these choices leads to consider the additional axis j as the expression of an ordered monoid (ℕ+, ≤). As previously shown a sums (coproduct) and a product are therefore entangled through the "fibration" within the folding, namely through the mathematical expressions of the zeta transformation of ℕ therefore $N(s+jt)=N(s)\times[0,T]$. In other works we can also defined $\vartheta(s)$ such as $\vartheta(s)=\{s + T_K\}\times_s N(s) \sim N(s) \times T_K$ (Figure 8) is both initial and final categorical object and authorizes the presence of a pulling back image, namely the reversibility of the "time" t. All along the scanning and the extension the s the mesh is preserved. The result is intuitive and suggests an extension of a ring for a module when the set $T \subset [0,\infty]$ is scanned, exactly as if, starting from an empty plane ℕ plane as the initial Memory, we have seed this initial memory by using a second "Memory" that grows

[5] Due to the domain initially compact for the definition of the complex variable, the replication using fibration, can isolate the different images of replication and therefore the fibration can create a layered structure, in such a way that each layer be clearly distinct than any others. This geometry opens the way for writing of the fibration as a Cartesian product.

up to be finally identified with ℕ as a replica of the initial one (Figure 8). Asymptotically we have created a "*Memory of Memory*", both being empty. In each location of the initial memory we create a replication to the first one, the whole set being nothing less than ℕ unique Memory. Asymptotically and whatever the value of s: ℕ × ℕ = ℕ; the result of this construction is a peculiar expression of the self-similarity of the set of integers[6].

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[6] Detail proof: s is extended according to $s + jm\tau_0 \cdot \tau_0$ authorizes the separation of the replication sheets. Let us assume that $\{m\}$ is the set of successive power of a figure π that means that the space ℕ is represented along an axis characterized by a norm π, the scanning being performed at scale τ_0. The axis can only be scanned one time, and thus the fiber matches the set $N(s)$ modulo the axis labeled π. The total measure of the exponential operator applied at each fiber gives the reference through the shifted zeta function $\zeta(s + jm\tau_0)$ with $\tau_m = m \times \tau_0$. But using the multiplicative formula of the zeta function we can write $\zeta(s + jm\tau_0) = \zeta(s)(1 - \pi^{-s})$. But $s \in K$, therefore $|\pi - s| \leq n^{-a}$ with $a > 0$ and thus, $|\zeta(s + j\tau_m) - \zeta(s)| < \varepsilon / 2 \forall s \in K$. Up to this step, $\tau_m \in [0, T]$ a discrete set countable and infinite. Obviously this set has a null density because $\lim_{T \to \infty}\left(\frac{1}{T}\right) mes\{\tau_m \in [0,T]\} = 0$ but we can consider a vicinity of τ_m, υ_m, such as $\upsilon \in [0, \infty[$ in ℝ to reach, $|\zeta(s + j\tau_m) - \zeta(s + j\tau)| < \varepsilon / 2 \forall s \in K$, and $\forall \tau \in \upsilon_m$ and therefore, $|\zeta(s + j\tau) - \zeta(s)| < \varepsilon \forall s \in K$. Existence of υ_m is ensured because zeta function is absolutely continuous in his realm of convergence. Let us recall that the integral values of m were obtained after rearranging the numbering corresponding to the isomorphism ℕ × ℕ ≃ ℕ. relation that means that the points in the complex plane no longer fit to the natural order related to the monoid $(N+ \leq)$ but another one. We are not disturbed because the important point is that these points of the complex plane form a discontinuous suite completely unbounded.

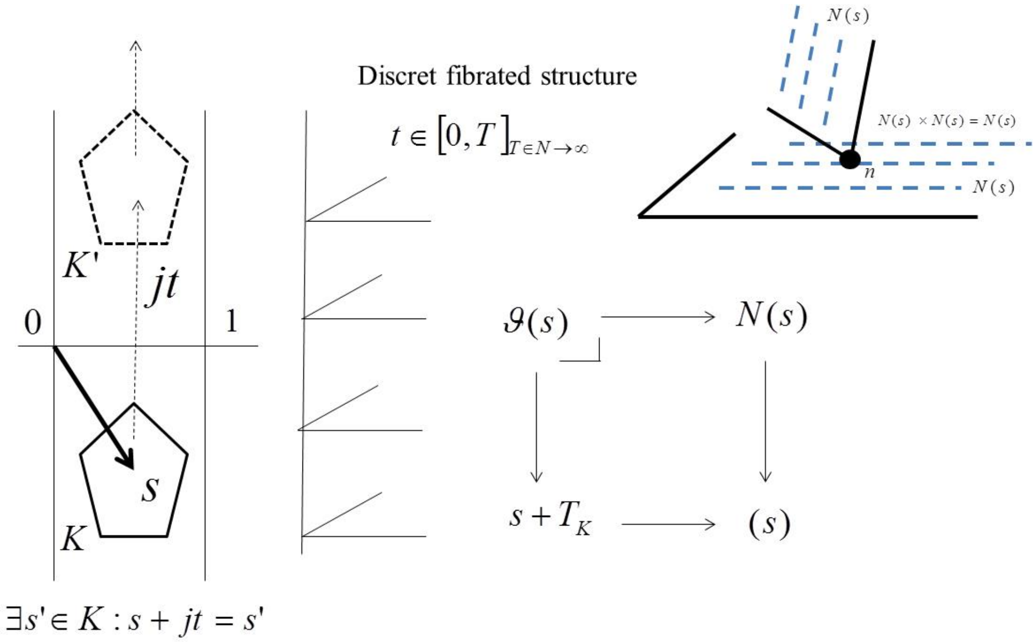

Figure 8. $N(s)$ – Voronin Fibrations: We see in this scheme, how a compact domain of the complex plane can lead to a discrete foliation along a complex axis adapted. We also observe how this quantification leads a composition giving birth to a "*memory of memory*" that remains nevertheless a unique ℕ "device of memorization", device solely constraint by the need of a new label of every point (characterized by a natural number). Considering the status of the infinity, the morphisms between both memorization devices lead to the simplest relationship of the self similarity $\mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$. Let us observe that this situation comes from the Voronin implementation of the functor: $s \to s + jt$.

Practically we observe that the order involved by the filling along the fiber can be linearly embedded in a copy of $N(s)$ gradually fulfilled to reach ℕ. The actualization of the infinity imposed through a compactification, plays a major role in the emergence of the self-similarity associated to the presence of fix points at the infinite scales. Everything occurs in practice as if we had built $N(s) \times N(s)$. But for reasons of self-similarity of ℕ, $N(s) \times N(s) \simeq N(s)$. This relation condenses the expression of a fundamental isomorphism and therefore the presence of a hidden internal functor in ℕ including functor between limit and co-limit. The isomorphism is equivalent to a new indexation of the axes (through the use of primes) that defines the n value also (Figure 8) in the space $\vartheta(s)$. According to this depiction, it is always possible for the axes corresponding to the second factor $[\times N(s)]$ to have sufficiently high ranking within the $\mathbb{N}$ internal order, this situation being able to split the set of memory, into two distinct sets. Indeed, we can write $n = \Pi_{p \in \wp} (p_i)^{r_i}$ and we can apply this equation to share the labeling of $n \in \mathbb{N}$. This operation being realized, the definition of zeta as the trace of the exponential operator upon each space and upon the common space can be obtained. Therefore, as zeta function may be defined both as a trace based on $\vartheta(s)$ and as a trace based on $N(s)$ both traces converging toward the trace on $\mathbb{N}$ at the limit $t \to \infty$; the Bagchi inequality naturally emerges not as a wrong analytic extension of the Voronin inequality, but from the self-similarity of the generic $\mathbb{N}$-support of his categorical proof according to the specific process used for construction (via Cartesian products and definition as a referential trace upon this (these) support(s)). Therefore the prohibition of the involvement of any analytic functions $f(s)$ characterized by the presence of a set of zero in the compact K (characteristics of zeta functions) required by the Voronin inequality, stays valid even if this constraint is able to be by-passed for the zeta functions characterized by the existence non trivial zeros in the compact field. The consequence of Bachi inequality (4) is the self-similarity of the zeta function and therefore the validity of the Riemann hypothesis that asserts analytically that the set of non-trivial zero of the zeta function $\zeta(s) = 0$ are located on the line given by $\alpha = 1/2$.

$$\forall \varepsilon > 0, \lim_{\inf} [\frac{1}{T} mes\{t \in [0,T]\} : \max | \zeta(s + jt) - \zeta(s) | < \varepsilon] > 0. \qquad (4)$$

D. Synthesis of a categorical approach of Riemann Hypothesis

Summary: The above approach of Riemann Hypothesis is based on a couple of idea.

The Riemann zeta function $\zeta(s)$ is nothing else but the measure of the exponential function computed on a discrete, topological, countable, and infinite space. The variable s, accounts for the scaling factor applied to this space. The double description of the zeta function either additive form (categorical co-product) or multiplicative form (categorical product) as dual functorial structure, highlights the fact that $\zeta(s)$ is naturally associated with a process of matching or algebraic equivalence. The forcing of the asymptotic coincidence, that fits a compactification implementation, and more generally the correspondence between the product and the categorical coproduct imposes the existence of a fix point (attractor) pointing out the s self-similar properties of the underlying Riemann manifold structure. Both aspects (i) the matching between product and coproduct and (ii) the self-similarity are used to validate the Bagchi's Inequality. Indeed the analytic realm is not necessarily the suitable field to manage easily the mathematical roots of the Riemann Hypothesis.

The main characteristics of s is given by the fact that s belongs to a compact field K. $\mathbb{C}$ being conventionally associated with a vector $2D$ in $\mathbb{R}$ space (finite vectorial dimension), according to the Borel-Lebegue theorem any compact field in $\mathbb{C}$ is a bounded and appears a close set. As shown above boundedness is essential for the reasoning. Indeed, choosing the parameter absolute value $|t|$ sufficiently high with respect to the diameter of $s : s \in K,$ the transition from s to $s + jt$ with $t \in T_k$ can allow the creation of translated compact K_T without overlapping along the axis jt. By iterating this operation a countable infinite number of times we obtain an infinite number of copies of K without any overlapping. From this geometrical point of view, but also in terms of category theory, this construction matches the construction of a Cartesian structure. This can also be considered as a shift of the referential in the theory of modules over rings and likewise in Galois Theory. This way of understanding which is the structural roots of Riemann Hypothesis leads to substitute to an additive operation too much intuitively given, a multiplicative operation performed from a Cartesian product of sets able to describe a fibrous structure expressed by the transform $s \to s + jt$.

A given variable s must be understood as a scaling factor associated with the space the set $\{n^{-s}\} = N(s)$. Due to the geometric representation of $\mathbb{N}$ of the set of natural numbers n whatever the value of $\Re(s)$ $\alpha \in \left[\frac{1}{2}, 1\right]$ any point of the topological space is entirely determined by an integer n. This space is intuitively interpreted as an initial memory without any information content. The operation we have described above implies the replication the space $N(s)$, namely the replication of the memory. Thus we create a "*memory of memory*" which, far from the intuition and due to the presence of the infinity within the sets considered, is actually not larger than the initial memory. This last memory must nevertheless be structured appropriately to the matter especially with a smart indexation of the sites of $\mathbb{N}$. The term in quotation marks is deliberately highlighted to emphasize the major role imparted to the concept of Monad, leading naturally to the concept of idempotent-Monad. This last underlies the concept of fix point (anchor set and attractor), properties well known of the experts in category theory. In this case the fundamental equation of fixed point is the simplest and most fundamental of all, namely the isomorphism given by $\mathbb{N} \times \mathbb{N} = \mathbb{N}$ to which any self-similar construction can be reduced. The overall space thus obtained, namely this new memory, can be reorganized so that all point in this totally disconnected space, or even any memory location either characterized unambiguously by an integer, that requires a new locations numbering process in memory. Given these main elements: $\zeta(s)$ function and its shifted dual $\zeta(s + jt)$ can be compared and after a careful analysis, we obtain for $s \in K$ and $t \in [0, \infty]$

$$|\zeta(s + jt) - \zeta(s)| \leq \varepsilon$$

namely the probe that zeta function is a self-similar function. The main consequence is the fact the conjecture of Riemann is true. The above categorical approach ensures that the distribution of non-trivial zeros of the function $\zeta(s)$ along the pure complex axis given by $\alpha = 1/2$ is mainly due to the self-similarity properties contained in the isomorphism $\mathbb{N} \times \mathbb{N} = \mathbb{N}$ set which exists *per se* (without any outside). Such isomorphism is in practice achieved through a self-similar structure, also named of fractal with d=2. There are currently multiplicities of ways for obtaining such structures, characterized by a dense and continuous set. It ensures that the construction and the partition are adjoint operations, the physical meaning of which is the steady state: namely the local reversibility of the "time" t. The reasoning explains why the main attempts to prove the Riemann conjecture by using tools, generally very sophisticated, of complex analysis have failed up to now. The justification of the conjecture may be based on the relevant ways used for changing the dialing of the partitions of subsets, for any infinite countable set of objects related by morphisms. The categorical approach clearly suggests the nature of the relevant concepts which must be involved in the reasoning: adjunction, monad, Kleisli category, memory etc. But are these arguments and its link with Riemann conjecture the right source of what is observed experimentally? Is this explanation sufficient to explain the rising of the entropy experimentally observed when self-similarity increased from d=1 to d=2, presumably when the unit time becomes irreversible? Even if the new Riemann approach unveils the roots of the long resistance of the analytic Riemann enigma, it does not give directly the answer to these questions. Nevertheless this new approach may help us to find the right perspective for answering.

V. Time Unit. Nature of irreversibility. Uncertainty *versus* Incompleteness

Much more than the concept of space, the status of the concept of time stays in the post-modern world, an open question. The Greek civilization distinguished three concepts: (i) *Aion*, the cyclical time able to testify of cosmic order; (ii) *Cronos*, the time of the history, of the tales, the time of the exploration and of the progress of knowledge; and finally (iii) *Kairos*, the metaphysical time, the time of the appropriate moment, the time of the changeover; we would say today the time of the Dirac's delta distribution. We will show that this classification is excellent but that the morphisms between these different concepts blur the boundaries between these notions. To achieve a control of nature, the Cartesian project has sliced brutally in this difficulty by reducing the time at a single mathematical parameter and Emmanuel Kant bypassed the epistemological difficulties associated with the new status of the temporality by asserting that space-time are only *a priori* categories. However the treatment of complex systems requires a critical deeper analysis giving to the time an opportunity to reborn as a fundamental factor of our relationship with the world [80]; renaissance fortunately authorized among other notions by the development of the advanced thermodynamics [81, 82, 83] physics of loops (as specific adjunction) [80, 84, 85], and fractals geometries (scaling morphims). But let us stay for a still short moment, with Riemann Hypothesis, and the mechanical concept of the time, accordingly with the Quantum Mechanics which is also the fount of non-commutative geometry and as shown above the puzzling issue of the Riemann Hypothesis [86, 87, 88, 89] (also based on the role of a residual phase angle).

The fibration of *N*-set according to the transformation $s \rightarrow s + jt$ leads naturally to prove the self-similarity of the function $\zeta(s)$, namely the set for which the procedures of construction *versus* partition comes from a functorial adjunction. Another way of highlighting the same properties this is to write some physical equilibrium of the approximation of $\zeta(s)$ via the equality $s + j(t + \Delta t) = s + j(t - \Delta t)$ extended linearly to $s + jt = s - jt$. The latter relationship is obviously associated with the Riemann conjecture since it is clear that if $\zeta(s) = 0$ leads also $\zeta(\bar{s}) = 0, \bar{s}$ being the complex of $\bar{s} = (1/2)j\theta$ if Riemann Hypothesis is true. The result of the above analysis leads to even identify $\zeta(1-\bar{s}) = \zeta(1-s) = 0$. Cross equality with θ reversible $\zeta(s) = \zeta(1-\bar{s})$ ensures the locking of the factor $\alpha = 1/2$. Thus, if the problem of the Riemann hypothesis is considered in the dynamical framework of exponantial, the time is split into two factors (i) the physical traditional *N*-time ($N + \leq$) coming from Fourier transform of the dynamics giving information within the scaling [21,22], namely *n* whose correlations determines α and (ii) the complex component of $s : \theta, t \in T_K$, scanning the whole set of the possible time-constants able to be meshed with the dynamics. In the framework of Riemann Hypothesis, the couple of parameters (n,θ) is able to flow reversibly in both direction from zero toward infinity or from infinity toward zero through a mere inversion of mathematical expression. But in this case the self-similar constraints with $d = 2$ therefore $\alpha = 1/2$ must be fulfilled to ensures both (i) the existence of a 2*D* fix point at infinities and (ii) the absence of influence of any "exterior factor" at the limits. Conversely, the reversibility of the time (through the parameterization of the geometry) becomes condition of the validity of Riemann hypothesis analyzed in the Exponantial framework. The relevance of the Riemann conjecture analyzed through the dynamics approach and especially through the new concept developed in paragraph 2, leads therefore to two implications

- The metric, namely the fractal dimension of the tree associated with the Riemann Manifold is given by $d = 2$ [34] and thus its self-similarity is similar to a structure Péano tree situation which excludes any "externality" for the set of the objects that meet the condition $\zeta(s) = 0$. There is no extrinsic environment to the fix point, which then, is not something like a "fence". This conclusion is similar to that of Edward Nelson in Quantum Mechanics [131], asserting that fractal dimension of the manifold underlying the Quantum Mechanics is $d = 2$ and that the rules which managed by the Quantum Mechanics find their origins in quantum fluctuations of a set of stable states. This intrinsic effect depends only of the overall set of states. There is not any hidden variable in Quantum Mechanics and its model is closed over itself.
- The sign of the "time variable", *t* and θ plays no role in the above approach of Bagchi's inequality and we can assert that this statement determines an implicit access to the main symmetry characterizing the Riemann Hypothesis: $\zeta(s) = \zeta(1-s) = 0$ and phase angle ψ=0 (Fig. 2 and Fig. 10). This property is used for the physical model of the different kind of mechanics.

Therefore, besides the standard engineering objects (whose steady state can be reduced to an asymptotic state provided that the control of the relaxation process toward an equilibrium, uses a reversible temporality, namely quadratic properties), the "objects" of Quantum Mechanics appear like a set of stable states controlling the exchanges on manifolds whose approximation by a three has 2 for fractal dimension and whose dynamics uses the standard reversible temporality of the mechanics. This specific use of the mechanical time explains obviously the links between quantum mechanics and stochastic zeta properties pointed among others by Snaith [17], Granville [18] and Hawking [56] but also by Connes for his designing the non-commutative geometry [87], but also more further, the link with the constraints fulfilled by the fitting with the Riemann Hypothesis [88]. But these constraints are very restrictive in terms of categorical analysis. As suggested in Figure 9, our capability of taking into account the geometrical irreversibility of the time by starting from the above categorical analysis opens a larger field imposing the status of the Peano's self-similarity (Riemann and Quantum Mechanics Conditions $d = 2$). Now we would like to show that, taking into account the direction of rotation in connection with the phase angle φ or ψ (Fig. 2 and Fig. 10), – the complication leading $\zeta(s) \neq \zeta(1-s)$–, changes the nature of physical objects concerned by the above analysis, namely by the algebras determined by the set of functors. These objects are much more complex than traditional objects handled by the engineering, and much more complex also than the objects of Quantum Mechanics, whose states, except for uncertainty, can be well defined. The complexity of these objects comes from extrinsic entropic factors (or anti-entropic), influencing the reversibility of the temporality itself. These factors are due to the peculiar field of self-similar metric $2 > d > 1$ of the tree associated to the manifolds of the dynamics. This tree cannot any longer be reduced to the sole Péano geometry, but in this case it is perforce embedded in a larger geometry than itself. This geometry which "contains" the fix point must play a dual function of environment, much more complex than a mere complementarity. Thus the incompleteness arises naturally from the shift of d. Nevertheless a large part of the theoretical arguing about the categorical foundations of Bagchi's inequality stays valid. It is only stretched when the use of s is analyzed through the couple of components α and θ (Figure 9). As shown by the experimental data, the zeta function continues to play a central role for structuring the dynamics. Part of the theoretical reasoning is then based on the functional relationships that characterize the zeta function $\zeta(s) = F[\zeta(1-s)]$ relation that explicitly involve the sign θ, thus the temporal variables and their singularities. In continuation of this analysis we would like to show that *incompleteness* revealed by the concept of expon**a**ntial must be strictly distinguished from the concept of *uncertainty* and that both concepts merge if $\alpha = 1/2$.

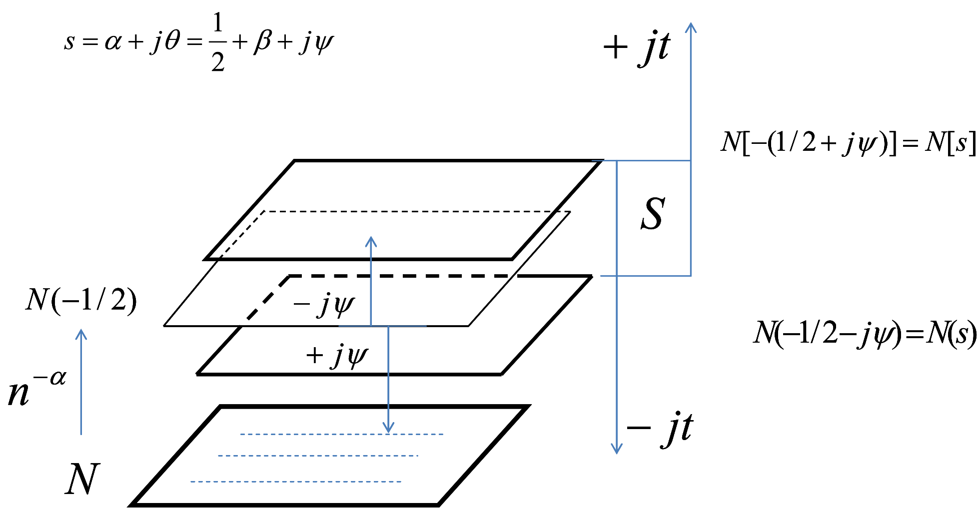

Figure 9. Taken into account of θ. Representation of the fibration if one takes into account the presence of a minimum value of the angle θ. This step in the initial approach deflects with respect to the assumptions of Voronin and Bagchi while remaining in the same axis of thought. Obviously, by the introduction of a temporal irreversibility, this step nevertheless changes the framework of the analysis and leaves aside the categorical objects obtained by starting from the Riemann hypothesis. Herein the constraint $\zeta(s) = 0$ is replaced by the functional relationships characterizing zeta, but $N(-1/2)$ stays helpful as reference.

The shift from exponential the analysis to zeta analysis is based on the embedding of the time into the complex field according to (n, θ). But via zeta this operation leads the components to entangle themselves because the factor of metric α (used to define the generic term of the series for the zeta function: $n^{-s} = n^{\alpha} \times n^{-j\theta}$) also defines at least the residual phase angle that must be used for locking the s-rotation at the boundary. The entanglement is notably based on the permutation of the sign of the rotation in the bulk and at boundary $\theta = \pm \psi_{\alpha} (\mathrm{mod} \mp 2\pi\nu)$. Internal correlations upon the dynamics implicitly create (or are forced by...) the geometry of the environment. The entanglement which is not obvious to be defined when the analysis is developing from $N(s)$ above analysis (Figure 9) becomes obvious by changing the representation (Figure 2 extended to Figure 10). In this case the ordered monoidal additive structure is associated with the Fourier transform of the usual physical time; the scale component is now given by factor $t + \theta$. The key point is the fact that, except in the constraint of Riemann hypothesis, the residual angle ψ_{α} (Fig. 2 and 10) cannot generally be reduced to zero. This tiny difference becomes significant for expanding the reasoning because the splitting of the complex part of the temporality into a couple of factors cannot ignore especially a possible difference between the respective signs of the terms of the couple t and θ. By starting from the physical formulation, this difference must be shown within the complex plane of the representation. The geometrical and analytical consequences of the physical constraint implemented through ψ_α, are reported in details in previous notes [21, 22, 33], and are summarized in the geometrical representation given in the Figure 10[7].

In practice, the initial categorical reasoning supporting the new approach of Riemann Hypothesis is preserved, with the exception of the following subtlety: the physical results already obtained suggest shifting the representation of the N self-similarity according to $N(s)^2 = N(s) \Rightarrow N(s)^{1/2} N(\bar{s})^{1/2} = N(s)$ that would be a layered structure if the sign of t is taken into account. Indeed if the fibration is reversed a trivial support is rebuilt and to understand the role of the sign of the fibration in its general form, we have to considered firstly the complex expression of the function s and secondly extension in the form $s = \alpha + j(\psi_\alpha + t)$ from an irreducible minimum basis $\psi_\alpha = \theta_{min}$. The efficiency of the representation of the figure 10 is due to the fact that it makes coherent the reasoning over the whole set of zeta functions $[\zeta(s), \zeta(1-\bar{s}); \zeta(1-s); \zeta(\bar{s})]$ which then appears, under the reserve of cross isomorphism, as a set of categorical co-limits. The nature of the

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[7] In affirming without risk of error that the value $\alpha = 1/2$ corresponds to constraint leading the trapping of a reversible time, we are led to examine all additional cases characterized by an irreversible complex component of the time. These cases lead to take into account the phase angle naturally associated with $\alpha \in \left[1, \frac{1}{2}\right[$. Referred to the exponantial dynamics (with a) the basic phase angle is given by $-\varphi = \frac{\pi}{2}(1-\alpha)$. However, the order of the axes in the complex plane being able to take into account the sign of the variable t it's suitable to choose as a reference axis of rotation t and θ the axis of symmetry forcing inversion of time to be neutral, namely $y - j = i^{1/2}$ then $\pm \Psi_\alpha = \mp \frac{\pi}{4}\left(\alpha - \frac{1}{2}\right)$ figure 9. Thus the sign of the phase angle induced by the metric α is inverse of the sign of rotation _ and if nonzero $\alpha > 1/2$. The geometry becomes non-commutative since it arises an irreducible angle at boundary. Throughout the realm given by $\alpha \in \left[1, \frac{1}{2}\right[$, the usual physical time remains as $n \in N(s)$, nevertheless $t \notin [0, \infty]$ but $\pm t \in [\mp\Psi_\alpha, \mp\infty]$.

isomorphism is well known: it is the functional relationship between $\zeta(s)$ and $\zeta(1-s)$ namely $\zeta(s) = 2^s \pi^{1-s} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$. This relations should base the extension of the categorical analysis according to the following hypothesis of closure: $N^{1/2}(s) = N^{\alpha/2}(s)N^{(1-\alpha)/2}(1-\bar{s})$ and $N^{\alpha/2}(\bar{s})N^{(1-\alpha)/2}(1-s)$. These expression appears here as assumptions.

While spectrum analysis attaches to cyclical time through the Fourier Transform (*Aion*) of the traditional physical time, the singularity at the boundary of the fibration, – singularity related to the metric of the fractal underlying structure, –, leads an irreversibility of the complex component (*Cronos*) of the hybrid time. So that, through any parameterization opportunity given by t, the mathematical parameter is not irreversible by itself. Which causes the dissymmetry is the positioning of the phase angle θ_{min} (implicitly determined by the factor α) which explicitly combines the causal and non-causal arcs within a complete dynamics that is simply an expon**a**ntial dynamics, namely the Aion correlations leads the geometrical embedding that leads itself the Cronos irreversibility. The key aspect is the fact that zeta Riemann function formalizes this features through properties based non longer on the expression of a dynamics but through the expression of the hyperbolic less action arc of circles, properties that have little to do – in the case of the non-causal arc-with the physical phenomena reducible to causal temporal dynamics. The non-causal effect might be considered as the *Kairos* component.

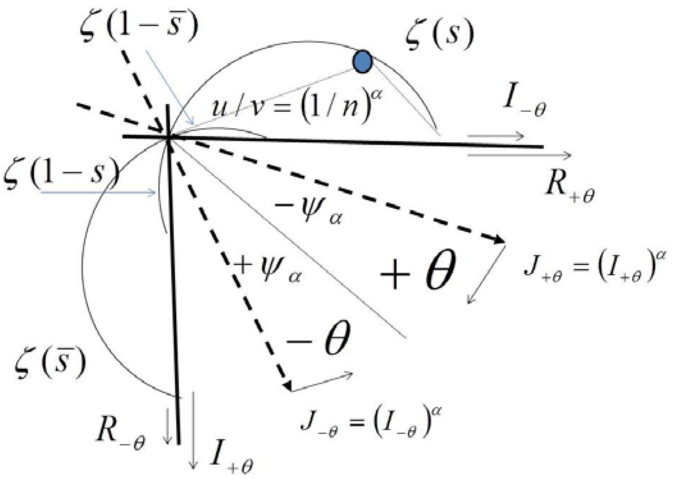

Figure 10. Other representation of the fibration based on the use of exponential schematic diagram. This last points out clearly the role of the sign of the time variable and highlights the singularity associated to the validation of the Riemann hypothesis, viewed as the case where time irreversibility disappears, namely a physical return to a mechanical model.

Thus one must distinguish two fields of reasoning: (i) A field in which the conditions of the Riemann hypothesis are met; The zeros of the zeta function defines the "physical objects" of this field (the zeros build indeed a topologically countably discrete infinite set) that also contains many hybrid other physical objects, reduced to the "transitions". (ii) A second field builds a realm where the constraints of the Riemann hypothesis are not met. In this field the relationship $\zeta(s) = 0$ which ensures the simplest completeness and replaces it by the functional relationship $\zeta(s) = 2^s \pi^{1-s} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ that also aims to ensure the closure an hybridation of the complex systems, and that brings implicitly with it the presence of the exponentiation operator. In fact this operator matches the introduction given in the paragraph 2 and the categorical approach given in paragraph 4 herein focused only on Riemann Hypothesis. If the hybrid systems respond according to the constraints given by the above general point of view, such a system will be henceforth named *Zeta Complex Systems*.

Now we have to consider a fundamental distinction between *uncertainty* (as seen in Quantum Mechanics) and *incompleteness*. For *Zeta Complex System*, *incompleteness* is associated to the hybridation that involves a relationship between a $\zeta(s)$ dynamics and its $\zeta(1-s)$ expon**a**ntial complement; This complement involves the appearance of a degree of freedom. These systems must be also characterized by an *uncertainty* which is related to the fact that any definition of data with an infinite precision is impossible because it would be necessary to use prime set to do this. In the context of physical experiments, absolute data must be approximated through a cut within the rational numbers. In addition in the framework of associative algebra (and therefore according to the need to express any physical situation into a language using words), any expression must have a upper limitation and a lower limitation (linked together) that bounds our experiments. Both co-limits are joined by a morphism. These limitations may appear through fundamental *uncertainties*. Both *uncertainties* and *incompleteness* are related to the irreversibility of time, but for *incompleteness* this irreversibility is extrinsic and appears like a degree of freedom, while for *uncertainty* this irreversibility is intrinsic and appears as a stochastic limitation.

Last precision: if the concept of *Aion* can be associated with the cyclical time and if *Crono*s can be associated with the time of rotation and of approximation (t and θ), the concept of *Kairos* can then be associated with the temporal singularity that modifies the flow of *Cronos* at its edge. This singularity contains a considerable amount of information which determines an overall dynamics. Through the position of its singularity in spaces *Kairos* can alone give birth to a new phenomenon; it is clearly associated with the concept of creation. We see here that if *Aion* and *Cronos* stabilize the course of things that run naturally, while *Kairos* plays the role of a maieutic. We assume that currency insures this duty of creation with regard to the economy.

VI. Currency as archetype of a Kairos Factor in Zeta-Complex Systems

The physics considers usually the behavior based on stable states of matter [91]. These states are handled as categories likely to exchange 'extensive' thermodynamic variables (able to be counted) according to algebraic protocols in geometrical and differential or discrete "geometries" [81, 82, 83]. These "geometries" can be expressed in terms of functions, in terms of distributions or in terms of groups of transformations. It is assumed that any real object is defined in space and in time. The linear or quadratic time-space coupling is a refinement that always involves that the time is nothing less than a reversible time. In this framework *Aion*, the time of the clock, is a parameter able to be eliminated of the model if the evolution of objects in time is such as the motion of an object can be localized with regard to the set of the others. The equations of physics are therefore very limited in number and mainly associated at differential forms fairly simple – even if the solutions of the equations cannot be found easily or that major uncertainties characterize them –. In this context the duality of the "extensive" thermodynamic variables (able to be counted), versus the "intensive" variables (able to lead a measure) is central and similarly, the role that plays the exponential functions and logarithms in the implementation of the rules coupling both entities. When physical systems are frustrated, the renormalization techniques [92] – whose legitimacy is not assured, but whose experimental effectiveness is established – replace, through scaling management, the standard optimization of the energy at a given level of scale to reach the emergence of stable states at higher scales of relevance. Step after step, using techniques very close to the methods used in the frame of distributions theory (test functions), the renormalization is able to smooth the divergences characterizing the singularities, for giving birth *in fine* to the physical laws observed macroscopically. So is it, for the scaling properties of the turbulence which transcend the complexity of the nonlinear Navier Stokes law who founds it; It is the same for Ising Model in which the renormalization techniques subsume the existential frustrations between entities distributed on discrete networks. Even if the processes are irreversible (turbulence, relaxation etc.), the main feature of physical objects involved in these processes (electron, ion, mass, etc) is the fact that the objects must be considered as existing (i) in a space and (ii) with a strictly reversible time unit. Unfortunately the objects of the "human sciences" do not fit these conditions and may exist only in abstract spaces and only in the framework of irreversible processes. These objects exist like temporal singularities which disappear if the flow of entropy is cancelled. It is the case for living structures. These objects require to be described in tales which are different of the "standard scientific tale". We assume that the main difference with classical physics is related to the concept of time characterized by the existence of a singularity at the boundary of the set of temporality (see above). This situation is mainly due to the substitution of the standard exponential by the concept of expon**a**ntial (with "a") defined in paragraph 2. It may be sum up into the notion of *Zeta Complex Systems*.

The shift represented by the replacing of traditional physical systems by *Zeta-Complex Systems*, whose econophysics would only be a component, is far from being admitted – as an epistemological requirement – by scientific institutions. Living systems, social systems, the science of idioms, behavioral systems etc seem to be able to escape yet for a long time to the rational and reductionist bulldozers of the scientific disciplines. The experts in charge of these systems, however, know they can rely henceforth on analog models based on numerical simulation tools [93]. Thus the agents based programming may represent either the dynamic exchange within a termite mound including its environment or may use the constraints of graph theory to control for instance to long term stability of the social networks. It is not without reason that companies like Google or Facebook hold a "fair value" so high in the international stock exchanges. Although numerical simulations be much more illustrative than explicatory, the numerical techniques and the treatment of the big data base their principles in the computer sciences whose foundations, though exceptionally rich, are mostly ignored of pure mathematicians; that why, these methodologies fail to identify guiding principles based upon fundamental mathematics and therefore restrict many results, to poor correlations. But it is a mistake. The mutual ignorance between pure mathematics and the computer sciences is not justified but seems to be a reality which herein must be taken into account. The economy which could be able to create this bridge seems to ignore this opportunity because it dreams itself, among the human sciences, as a methodology able to reach international recognition as a "pure science" based on mathematics, but, what kind of mathematics? The design *ex abrupto* of price of the Sweden Bank in economic sciences in memory of Alfred Nobel [94] is the proof of the reality of this dream that obviously concerns its sister: the financial techniques. At looking closer the economic models, we can show that they can be seen within their academic pillars as mostly based on optimization techniques, partial differential equations and variational computing, all methods able to be reduced to engineering sciences, although, unlike the in physics, the absence of boundary conditions only allows of giving trends and asymptotical developments (note that this situation opens the use of the very advanced tropical geometry) and not any absolute values referring to equilibrium of states (clearly fantasized in spite of the tight deployment of many mathematical tools) [95, 96, 97, 98]. The finance, which is counterpoint of the economy, is under the control of all the intricacies of the probability theory and the stochastic risk models are expressed by non-linear equations [99, 100], whose an example is given by the Black and Sholes equation, a couple of experts who have tested *in vivo* the relevance of their martingale (based on the Fokker-Planck model of diffusion) by going bankrupt. For non-linear coupling reasons, none reliable forecast can be given by basing an arguing on such a models and still less, any eventual replication of experiences due to the complexity of boundary conditions coupled with the bulk of the system. Note that this lack of reproducibility is however paradoxically implemented in such models by involving the reversible physical time variable, *Aion*, cyclic time associated with the Fourier transform of dynamics concerned. Virtually, the mathematics for engineers, through extensively used, seems powerless to give status of science to economic expertise which nevertheless wishes so much this recognition. However, in the framework of a new epistemological point of view [101, 15] and as shown aside by the approach of the Riemann conjecture that we followed, the support of the scientific reasoning in economy could come – not necessarily from pure mathematics – but from the Computer Science as a science of the representations of the languages and of the morphisms on the abstract and living objects. Our hypothesis is the following: to be relevant the economy must incorporate both (i) a strong coupling with the financial aspects and (ii) the influence of the fractality as suggested by Benoit Mandelbrot and his followers [102, 103, 104, 105, 106, 107, 108, 109, 110][8]. If this is the case the zeta-complex analysis could be able to meet the characteristics of the self-reference and self-organization of economic systems and therefore attribute a consistence to the idea of incompleteness of economics issues considered *per se*. If such a program had to be implemented in the future, the notion of duality between economy and currency should be seriously considered for defining what an economic object is. This last being then mathematically hybrid (economy and finance together) should be considered as a monad and modeled accordingly. The reduction into a couple of distinct categories must then be regarded like the symptom of a pathological state of the economy or as a mistake of model. In advanced model suggested above, the duality is formally in resonance via a general self-similarity, with the properties of the zeta function, the link between economics and finance being then capable of being expressed by means of functional relationships over zeta. Continuing the analysis, if the economy appears as the causal process of exchange in hyperbolic environment, currency must then appear as the vector of an underlying strategy based on the perception of a horizon whose function – as dual of the exchange of thermodynamic extensity –, is equivalent to a Cohen forcing or a Kan extension. This is precisely the traditional role of the currency for facing the risks of projection into the

[8] For example, anomalies in the measurement of the efficiency of the markets, implies the need to abandon the Gaussian forms of efficiency, expressions based on averaging and variances of data and on abstract models of prices (stationarity, independence of steps, finite marginal variances etc.). This representation of the efficiency is not relevant with regard to random Pareto data. The use of extended stochastic models, implementing for instance stable Levy processes and fractal structures is able solve a part of the observed mismatching. For instance the H-efficiency may replace the 1/2 efficiency associated with Gaussian processes. Nevertheless this small step in the economic thinking appears as a not huge in terms of paradigms that still govern the spirits in the global stock exchanges and on the financial markets.

future. The posture of confidence in a future state must obviously be based on an excellent knowledge of the causal-component of the economy seen herein in a dynamic context given by an expon**a**ntiel (with "a"). The perfect zeta-morphism between components, (i) economical set of exchanges and (ii) the amount and the geographical perimeter of currencies must give to the economic systems the stability of the healthy living systems, looking to the future. The framework here described (two complex torus connected together through a singularity [111]) is perfectly bounded and for this reason we suggest to attribute a Zeta-Complex Structure to the model of economics in which zeta function takes the role of an integral scaling constant. In the template given, the aim of the currency is to close any exchange of goods within a open hyperbolic environment (therefore all dynamics are non-linear and associated to $\zeta(s)$) by a debt exchange expressed in monetary terms in the complementary hyperbolic space associated to $\zeta(1-s)$. In the framework of category theory, both components of the duality can be expressed in terms of Kan extension. In this case the local economical balance at each levels of the economy should be understood in terms of categorical adjunction and not simply as trade flows balanced by monetary equivalents, situation that forgets the functorial difference between assets and currency. Look at the economy in categorical terms according to the above Riemann Hypothesis approach (therefore by including the complex analysis) amounts to replace the balance of these flows, by the matching of the inverse flows according to a categorical adjunction: an arithmetic ordered structure (expressed in monetary terms) and a partial order structure (expressed in terms of fungible assets, set of singularities). Such a point of view relaxes the financial constraint, operation which is justified (i) by the role of singularities (dynamic divergence) (ii) the joint role of self-similarity, namely the correlations existing between the different levels of the economy (renormalization) and (iii) by the degree of freedom for looking toward the future as an expression of the underlying almost sure hyperbolic geometry of the dynamics. The concept of monetary equivalence is then expressed through the functional relationship characterizing $\zeta(s)$ and $\zeta(1-s)$. It remains to distinguish in this framework the states of economic stagnation of the states of growth (the equilibrium being herein without any usual meaning).

The following general analysis led to distinguish between the set of "steady economical objects" and the set of instable situations. Consider first the steady states. They are by definition given by the statements: $\zeta(s) = 0$ and therefore the Riemann constraint $\alpha = H = 1/2$ has to be fulfilled. The associated space-time hyperbolic relationship is therefore quadratic, each behavior must be characterized through a velocity and an energy whose dimensional equation is respectively given by $[Lt^{-1}]$ and $[L^2t^{-2}]$. The consequence, as it was seen above, is that the time considered is nothing other than the *Aion*, the overall time irreversibility rising only through the action characterized by Gaussian correlations including the time: dimension $[L^2t^{-1}]$. Under these conditions the currency becomes a fungible commodity. It becomes able of being swapped against itself (example buying of own shares for increasing the stock exchange value of a company) and the anti-entropic feature disappears with the emerging symmetry $\Psi_{1/2} = 0$. According with the dimensional equation of the action, the sole order then able of being established is a probabilistic order. The system becomes Markovian and any arbitration is no longer determined by the long term perspectives, or the common interests of the community but in the immediacy of local interests of brokers. We assume that this situation is the one that might occur in the OECD countries if the current evolution goes on. The "financialization" of the economy is then characterized with a decrease of the alpha value non Gaussian efficiency $\alpha = 1/2$ toward $\alpha = 1/2$ namely a Gaussian efficiency with the disappearance of all the correlations except the probabilistic ones. The incompleteness required for thinking the future, is then reduced to an uncertainty, and is turned into nothing less than an optical illusion, the anti-entropic feeling being like the reflection in a mirror of pseudo-objectivity. The individualism becomes the rule. The comparison with Quantum Mechanics that fulfills the same Riemann constraints suggests distinguishing fermions (for fungible species) and bosons (when the vectors of morphisms are themselves particles). The Kan extension model for the money, – extended within a mirror model when the currency becomes fungible –, could also explain why the money does not behave with regard to the increasing of prices as a mere commodity. The demand for money is indeed known as being a covariant variable in terms of prices while the commodities respond to the price by fitting contravariant rules. In pathological situation $\alpha = 1/2$, even hidden during a certain period, the currency undergoes the same type of obsolescence than fungible goods, and the financial instability arises with the risk of collapsing of its rate. [K] The current social risk provoked by the instability must be associated at a pathological nature of a situation of symmetry that, reducing the currency into a commodity, yields both, Mathieu effect and the Mass Effect.

The Matthew effect and the associated disappearing of the social middle class have a scientific precise basis in terms of exchange of entropy. It is known for instance that our planet has been able to give birth to the human being and to develop civilizations not just because the sun is shining but because the planet supplies to the universe that surrounds it, more entropy than it receives. This uniqueness is what we could call the albedo effect. The planet conversion displaces the energy received since the field of ultraviolet toward the field of infrared radiation. Still with the notation used hitherto we can write $n_{in} = \omega_{in}\tau_{in}$ and $n_{out} = \omega_{out}\tau_{out}$. Assumption of thermal equilibrium requires for the survival of the species, $\omega_{in} > \omega_{out} \Rightarrow : \tau_{in} < \tau_{out}$. In other words our civilization depends on our ability to create slow processes by starting from fast process therefore on our capability to consider the shifting of the daily issues into long term projects. The purpose of currency is exactly to realize this conversion in the economic area. More precisely the aim is to use the hyperbolic metric of social and economic common space, namely s in our model to ensure this shift, therefore $n_{in} = \omega_{in} 1_{in} \Rightarrow (n_{out})^{\alpha} = (\omega_{out}\tau_{out})^{\alpha}$. According to this formulation we see that it is rather easy to operate a leverage in a framework of a collective organization, by increasing through the factor alpha the anti-entropic effects, while reducing the power involved in the system. This is how civilizations were able to develop in small communities. Thereby the creativity also was able to be very high even in communities with limited resources. But we have to point out that there is a perverse effect to this efficiency: it is to only use the leverage without any reduction of the incoming power and even with an increasing. In this case the communities ensure sole the contribution to the reduction and the incoming power goes on growing. In these circumstances it is impossible to favor the long term stability because the scaling stability is broken and renormalization is put in failure. $\tau_{in} > \tau_{out}$ increases the incoming power under the condition of increasing all inequalities (hierarchy), shaped by the curvature the hyperbolic social manifold. The extreme pathological situation is reached when the Riemann conditions are met. $\alpha = 1/2$, situation that is characterized by the transformation of a market economy regulated into a casino (random control) open for very few people. The consequences are a local dissipation of entropy into the single "social chance". This evolution is the movement that is ongoing and which the increase of the global instability is a mere signal. The Matthew effect takes herein all its meaning. The human being becomes a commodity. The currency is then exchanged on the base of a Ponzi scheme, the upper classes increasing their level of money accumulation to the detriment of the middle class that, disappearing, leads the disappearance of the civilization.

The mass effect amplifies the Matthew effect. The inability of thinking the future by handling the sole intensities namely the monoids (N ×), led the human society to defer its choice on the local intensities measure given by the monoids (N+ ≤) thus on quantitative criteria and total order structures (quality planning, lean management, cost killing, pole of excellence etc.). The result of this choice is a race for size and implementation of algebraic order social structures, flat ontology and reference to the chance. The arguments used to prove the conjecture of Riemann take their social meaning, the numerical ordering of the human societies, their ranking becomes binding each social group occupying a defined box and the exchanges between social groups being then "essentialized" and determined by their respective "qualities" expressed in quantitative terms (efficiency, competition, ranking, etc). The relationship between each group is then controlled by the chance or the social determinism. As in physics, the time required to describe this society becomes reversible and the future disappears as read on a tag in Lyon: "*because of general indifference tomorrow will have to be canceled*". Such is the situation traced by the perspective of current evolutions of the economy, the race to an "excellence" that should lead to an uncontrollable increase of the social entropy production (expressed *in fine* by the war) on the planet.

According to a more optimistic and formally richer, perspective but meanwhile more complex, the incompleteness coupled with irreversibility of time appears as the major characteristic of the *Zeta Complex Systems*. The complexity arises from the coupling of the set of the trace (full) of the exponentiation functors of action increased by all the morphisms that structure the categorical objects of the economy. The latter set involves in fact the irreversibility of time parameterizing functions which express the dynamic nature of objects on which the functors operate. The new approach of the Riemann Hypothesis reduces the point of view but offers in the same step the capability of thinking what occurs around the singularity of the set of primes point of view highlighted by Riemann in 1859. This extension is a fantastic chance for understanding the difference between a physical object and a complex hybrid object and for avoiding to the econo-physics the entrance in a dead end. The categorical analysis led to confirm the anticipation of Ilia Prigogine: every flows of extensity and / or field variable affecting a locally compact space causes necessarily the emergence of an organization almost surely hyperbolic and transiently associated to the flow of a time that defines an interior and an outer. The internal flows of exchange cannot exist alone. They are coupled with something more larger than them, which sketches an horizon...The role of the currency in the economy is to create this sketch and the categorical approach of the Riemann Hypothesis, and its consequences including the irreversibility of time if it was taken into account, might help to improve our point of view about the increasing complexity of our economy and the dangers incurred if we continue to reduce the econo-physics to statistical rules or to engineering rules.

VII. Conclusion

Based on the theory of morphisms, the proposed approach of the Riemann Hypothesis opens a perspective that goes beyond the sole mathematical perspective. The horizon opens to the status of the physical objects, addresses the question of the reality and of the beings in the world. Certainly, the conjecture reinforces the very special status of science in this questioning and gives a clear basis to the theoretical foundations especially to the Quantum Mechanics in relation with the set of primes, but in doing so it draws the outlines of a broader problem deserved to be explored: the problem of *Zeta Complex Systems*, as new field for sciences, certainly among others attached to Quantum Mechanics, but more broadly, giving birth to the science of hybrid and of open objects. Among these objects we have to count, the econo-physics, arts, social systems, etc. As it is shown in the chapter 2, the engineering data had already shown us the need for a thorough exploration of its hybrid objects whose Ferdinand Gonseth had already noticed the epistemological importance [60, 61]. Through the concept of initial and final infinite memory we understand in the chapter 3, why the Computer Sciences may help us to understand the refinement required for managing the labeling of the sites in an infinite memory and why this subtlety offers a new point of view on both zeta function and Riemann Hypothesis, especially when such a memory is infinitely duplicated through “fibrations”. If we match mathematical foundations of self-similarity and the physical point of view associated to the concept of expon**a**ntial as the basis of a trip in fractal structure, this infinite duplication shed incidentally light on the requirement of taking into account in physics the exploration of each branching point of the tree associated with a fractal structure and to keep in memory that the tree is only an approximation of self-similar Riemann manifolds. This requirement introduces naturally the irreversibility of time and therefore something like a new non extensive thermodynamics [81, 82]. The "hybridation" issues arise also naturally according to the question of the referential needed to manage any irreversible representation of our world and therefore any action in this world. The disappearance of this need in Quantum Mechanics is a particularity in the framework of the zeta complexity. It is due to the reversibility of the time used in mechanics and therefore to the additional symmetries thus introduced. In the general case the symmetries are not present and we have to consider the "scientific objects" by starting from internal and external morphisms. This point of view which takes into account the incompleteness of any representation leads naturally to the hybridization of new scientific objects. The links between economy, as an infinite network of exchanges of fungible goods, and finance, as a horizon of representation of the confidence necessary for these exchanges (and not as pension adjustment between commodities) illustrate in a particular case, this hybridization. It is essential to remember that the zeta function plays then the role of an overall regulator. Beyond the only probabilistic considerations or fantasized equilibria, the tools based on zeta function create finally new ways of thinking the complexity and holistic behavior. On this new trail, the concept of space-time finds a performative status, and loses its Kantian standing. Obviously, zeta function becomes a new tool for analysis. The function zeta contributes to set up a universal process of linearization, thus a construction of integral causality for hybrid sets of objects. But this causality is not local, as usually for econo-physics reasoning, but as a process of approximation, it is "integral" in the mathematical meaning. This integral aspect plays probably a major role in the creative capabilities of the brain and in percolation processes (revolution, protest polling, efficient advertising, burst of violence etc). The categorical analysis of the Riemann Hypothesis – based on the singular countable infinity and memory space but also multi-scale, built on the logarithms of the prime numbers $N(s)$ –, shows that the space holds a dual role very similar to the role that plays Fourier' space with regard to the theory of distributions. In this context the zeta function, as the trace of $N(s)$ construction can reference via the theorem of Voronin, any complex transformation. This status grants the zeta function of a role of an implicit pilot (may be of damping properties). It is able to play a role in any types of dynamics developed in complex systems. The reader can measure at this step the importance of this function including for the tomorrow engineering. May be, can he also measure the epistemological importance through the metaphysical analysis of the currency pointed out below.

Indeed it can be found in very deep literary work of Jean Louis Schefer titled "*The time of which I am the hypothesis*", the following analysis that illustrates the consequences of the previous analysis. He wrote what I quote at length, in an English free translation: "social science, anthropology, psychology, medicine have been without doubt, tales that assume a primary conception of time as a continuous breakable oriented set of irreversibility (order of our mathematical language), that is to say a time of nature, even though the human being is taking the place of nature and even if he has the control of its own laws. The idea of evolution is inherent at a knowledge organization and in this case the idea of decline, of degeneration or growth appears naturally. The underlying model might be the one that best represents the history of the currency (*dixit*), namely a medium of exchange whose level is based upon a certifying guarantor, a referential value (standard gold for example) an extension of "trade durations" that values their continued use in trade. The currency is a language without any meaning other than exchanges that constitute it. So that reduced to its concept the currency is like a language: it is the first sign which constitutes the reality of equivalences values of all kinds (in categorical framework we would say that the money thus guarantees the presence of isomorphism functors econophysics). It is not a thing but a transition between things; the means of their estimates and their valuation (as the measuring instrument in physics). This is probably what Nicolas Oresme according to the meaning given in his book, "*The origin of nature and the transfer of currencies"*, regards as the developments of the currencies: theory by which currency, as social pact, can circulate as a general referent (first idea that characterizes universals of Leibniz); it is important to note that the monetary theory of Nicolas Oresme is contemporary with the disputation about on the transformation of the substances of the bread, religiously devoted raised as the body of Christ : on how this mutation happens, namely as a definition of the substance. If the currency does not circulate, the death appears. Society cannot rebuild on its guarantor, the regular reference that produces its standard of exchanges. We see in the medieval fables the holy wafer acquired at gold price (and also for the symbolic price of the 30 deniers for which Judah sells the body of Christ): these purchases are sacrilegious because they are legally absurd. We still measure how Latin theology and canonical law are a continued development of Roman law, mainly commercial in nature. The consecrated host is not a fungible species, namely it is unique: it may not be subject of exchange (*it is just morphism*). What became of the body through the sacrament (transubstantiation) has not any reality as a thing and this being is invisible because it is unable to be traded. This extraordinary performance – whose attempt explains the closing work of theoretical paradoxes associated with the concept of proof in scholastic thought –, is associated with the birth of the abstract scientific objects (mathematical logic, analytical geometry and algebra); this work started from an analysis of a ratio: the one of the "quality" *versus* the "quantity", ratio based upon an unknown, the problematic definition of pure quality that even now remains a mystery (the one of the rationality). The concept of "body" has a relevant definition only as exchange value. This is precisely the impasse of a scholastic (*and may be modern*) definition of computable values. They are computable if they are fungible, namely being able to be represented by other bodies. The only way to overcome this issue was to integrate this value, as pure *aporia*, to a definition of time, specifically as a language and a grammar, as "vectorialization" and tripartite definition (third term being included); this time included in a calculation of profitability for any the social phenomena, reduced to a summary of exchanges (fluidity, loss, development of equivalence, hoarding etc.): This time must be computed from the only framework of exchanges and evaluations. The present (*represented in science by the concept of velocity*) is the time of enjoyment which preempts the future, namely rebuilt the feature that substantiates the ancient fatality attached to a representation of cyclical time (Aion). But the invention of Humanities (...) has precisely virtualized the past *(in fact the time itself*) replaced by considerations (as a basis for computation) of essentially defective nature: the memory; psychic memory, organic and generic memory, social memory ... These remnant-times are fictions; These fictions are in their beings and in their purposes theoretical or therapeutic, namely of a monetary nature: it records of something essentially vagabond, an order of ranking, namely something like sequential systems. This specific financial time is not experienced by the being, except under the constraints of society. The therapeutic efficiency built on this time, does not exist, its dream is computed, let us say, with its defects". It is precisely these defects which are taken into account in partial order, and it is these defects that must be cancelled to rebuild causality, by matching this partial order with a total order ... performance that gives birth to the geometrical self-similarity of our living world.

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