CONDITION FOR EXISTENCE & UNIQUENESS OF IRR – DESCARTES RULE OF SIGNS & OTHER ISSUES

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Abstract. *NPV and IRR are essential components of Corporate Finance and are used for evaluating financial viability of investment projects. However, some serious issues have been raised about existence and uniqueness of IRR. While exploring such issues, we work out the necessary and sufficient condition for existence and uniqueness of IRR for any generalized scenario. Moreover, this condition is fulfilled by the projects considered by practicing managers. This explains why IRR delivers and is so popular among practicing managers.*

Further, certain applications of the Descartes Rule of Signs to estimate the number of IRRs are found to be erroneous. The correct approach in this regard is outlined, and it gives results consistent with our findings.

Our analysis about the ranking of mutually exclusive projects reveals that, as a technique, IRR may claim a preference in this regard.

Keywords: *Condition for Existence of IRR; Condition for Uniqueness of IRR; Necessary Condition; Sufficient Condition; Descartes Rule of Signs; Mutually Exclusive Projects*

Introduction: Net Present Value (NPV) and Internal Rate of Return (IRR) provide the two most popular techniques for evaluating financial viability of investment projects. Both are based on the DCF method and assume a flat yield curve.

Studies reveal that large companies predominantly select IRR as the primary method for project evaluation[1,2,3]. However, some researchers have pointed out a few pitfalls arising out of non-existence of IRR in certain situations and existence of multiple IRRs in some other cases[1,2]. We shall explore these scenarios in detail and try to understand why practicing managers find IRR so useful for their purpose.

The analysis reveals some merits of IRR and clarifies why this method delivers. Most interestingly, we are able to derive the necessary and sufficient condition for existence and uniqueness of IRR for any

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generalized scenario in a compact form. We also indicate the proper way to apply the Descartes Rule of Signs to problems of IRR and how this exactly corroborates our findings. This further establishes the internal consistency of our workings beyond all doubts. While doing this, we constantly keep track of the possible reason why practicing managers may find the IRR method of Capital budgeting most acceptable.

Obviously, the IRR is to be viewed as a physical object and not merely as a mathematical entity capable of assuming any possible value – real, positive or real, negative or complex. Only the physically acceptable, i.e. real and positive solutions are candidates for IRR of any investment project.

Existence of a unique IRR would demand existence of only one real, positive solution for the equation involving "*r*", the discounting rate. Other roots of "*r*" satisfying the equation, viz. the negative, real as well as the complex roots, are not physically acceptable and need to be ignored. For instance, for a project with a time horizon of 5 years, the equation in *r* will have a term containing r^5 (highest power of *r*). Consequently, there will be 5 roots for *r*. If only one of these 5 roots happens to be real and positive, then a unique IRR exists. Absence of any positive, real root for r will mean non-existence of IRR. On the other hand, existence of more than one positive, real root will imply existence of multiple IRRs. Every case of unique IRR will naturally be associated with the existence of four unacceptable roots for *r* (either complex or real and negative roots). These simply need to be ignored.

Analysis: At the outset we recognize NPV and IRR as the two most popular Capital Budgeting techniques used for evaluating the financial viability of investment projects. We intend to focus here on the financial viability of investment proposals that may be expected to be taken up by practicing managers for evaluation.

As mentioned above, a few scenarios have been considered by some researchers to illustrate that existence of IRR may not always be guaranteed and multiple IRRs may exist in certain situations. A systematic account of such cases has been provided by Andrew W. Lo[1]. We have examined such scenarios seriatim in the following sections to explore whether there can be some scope to accommodate these scenarios within a single consistent framework.

The process of application of the Descartes Rule of Signs to estimate the number of IRRs for any given scenario is also examined and the correct approach indicated. As expected, proper application of the rule yields outcomes compatible with our findings.

With this prelude, we start discussing some of the scenarios concerning possible non-existence of IRR and possible existence of multiple IRRs. In line with our professed goal, our discussions henceforth will focus only on investment projects.

(A) Non-existence of IRR[1,2]

	CF_0	CF_1	CF_2
Project1	–105	250	–150

No IRR exists for this project.

It is important to note here that the sum of all the cash flows after the initial cash outflow is not adequate to cover the original outflow (250 – 150 = 100, which is less than 105).

In this context, we would like to develop our line of analysis from some simple cash flow patterns before working out a generalized condition for existence of IRR. All along, we shall also try to identify why the IRR technique may not pose problems for the practicing managers.

(i) We start with an investment proposal where an initial outflow of $P0$[i.e. a cash flow of (–) $P0$] is followed by a single inflow of after one year. Then $P0\ (1+r) = A1$, where r is the IRR. Or, $(1+r) = a1$, where $a1 = A1\ /\ P0$. Or, $r = -1 + a1$. So, a real, positive (non zero) solution for r can exist only if $a1 > 1$, i.e. $A1 > P0$. In other words, a physically acceptable solution for IRR exists only if the cash inflow at the end of one year exceeds the magnitude of the initial cash outflow. Further, this solution is unique because there is only one solution for r.

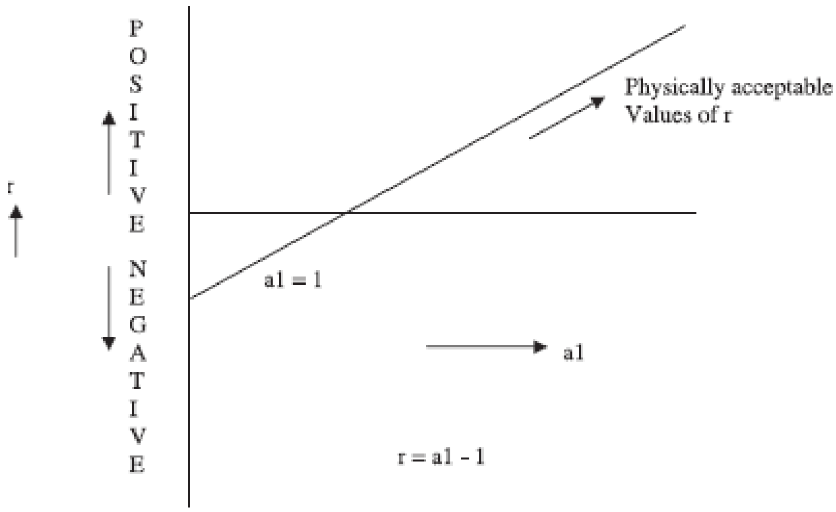


Figura 1.

(ii) Similarly, if there is only one inflow of $A2$ at the end of 2 years following an initial outflow of $P0$, then we have two solutions for r viz. $r = -1 + \sqrt{a2}$ and $r = -1 - \sqrt{a2}$, where $a2 = A2/P0$ and $\sqrt{a2}$ represents the positive square root of $a2$. Here, one solution of r, i.e. $r = -1 - \sqrt{a2}$ is clearly not physically acceptable because it cannot have a positive real value. The other solution is physically acceptable only if $a2 > 1$ i.e. $A2 > P0$. Thus, a unique physically acceptable solution for IRR exists if and only if $A2 > P0$.

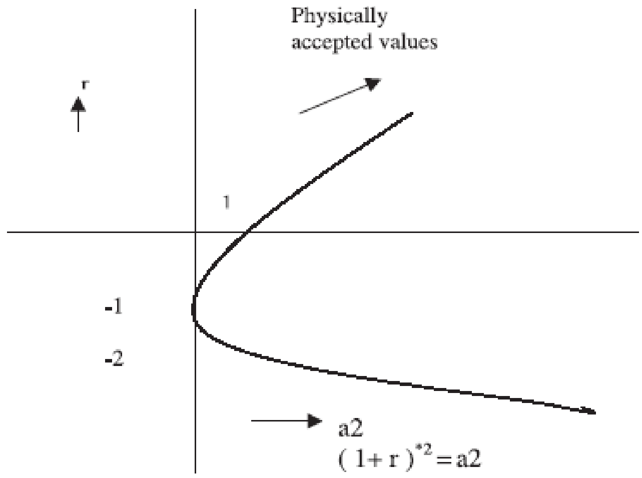


Figura 2.

(iii) Now we consider a scenario where the initial outflow $P0$ is followed by only one inflow $A3$ at the end of year 3. In such a situation, $P0 (1+r)^3 = A3$. Or, $(1+r)^3 = a3$, where $a3 = A3 / P0$.

The Internal Rate of Return will thus be determined by the equation $(1+r)^3 = a3$. rcan have a real and positive value only if $a3 > 1$, i.e. $A3 > Po$. The value of the inflow must therefore exceed the amount of the initial outflow. If α, β and γ are the roots of the above equation, then $(r-\alpha)(r-\beta)(r-\gamma) = 0$ or, $r^3 - r^2(\alpha + \beta + \gamma) + r(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma = 0$.

Comparing with the equation $r^3 + 3r^2 + 3r + 1 = a3$, or $r^3 + 3r^2 + 3r + + (1 - a3) = 0$, we can write down: $\alpha + \beta + \gamma = -3$; $\alpha\beta + \beta\gamma + \gamma\alpha = 3$; and $\alpha\beta\gamma = a3 - 1$. Since $\alpha + \beta + \gamma = -$ ve, all 3 roots cannot obviously be positive. But $\alpha\beta\gamma = +$ ve (for $a3>1$). Hence there would be either two negative roots or two complex (complex conjugate of each other) roots. There is thus only one physically acceptable solution for r that is practically interesting. In other words, a unique IRR exists in this case if and only if $a3 > 1$ i.e. $A3 > P0$. The other 2 roots for r are either real, negative or complex (complex conjugate of each other) and need to be ignored. For $a3 < 1$, however, either no IRR exists or two IRRs exist.

Each of the three cases discussed above has a unique, physically acceptable solution for IRR under a given condition. The condition basically warrants that the subsequent inflow, only one in number in each of the above cases, must exceed the initial outflow.

(iv) We now consider a more generalized investment scenario in which a single outflow is followed by a number of inflows and examine whether it modifies the condition for existence of IRR.

Let the cash flow pattern be like: (–) 100, 10, 20, 20, 25, *X*. This is a case where a single outflow of 100 units is followed by five successive annual inflows of 10, 20, 20, 25 and *X* units. The last inflow has not been specified but kept open so that we can examine the position for different values of *X*.

(a) If $X< 25$, IRR does not exist (no positive real solution for *r*).

(b) If $X = 25$, IRR = 0.

(c) If $X >25$, a unique IRR exists.

From the foregoing, we note that a unique IRR exists in case $[a1+a2+a3+a4+a5] > 1$. On the other hand, no acceptable solution for IRR exists if $[a1+a2+a3+a4+a5] < 1$ and IRR = 0 if $[a1+a2+a3+a4+a5] = 1$. In other words, the condition for existence of an IRR is now $[a1+a2+a3+a4+a5] > 1$. If the future cash inflows are spread over a longer period, say 10 years, the condition for existence of a IRR is modified to $[a1+a2+a3+a4+a5+ a6+a7+a8+a9+a10] > 1$. Thus, the sum of all future cash inflows needs to exceed the magnitude of the initial outflow in order that an IRR exists.

In order to accommodate a more complex investment scenario involving possible outflow(s) in future (nonconventional cash flows), we may work out, in the same way, a generalized condition for existence of an IRR of the following form: $\Sigma\, ai> 1$, where $ai = Ai\,/P0$, Ai being the cash flow at the end of i years, and the summation runs over the entire period of cash flows from the project under consideration.

Let us now analyze the most general case in order to explore whether the relationship between the existence of IRR and the value of $\Sigma\, ai$ is just a matter of chance or there is an underlying mechanism driving this relationship. The equation for IRR in such a general case would be: $P0 = \sum Ai\ /\ (1+r)^i$, where *Ai* represents the cash inflow at the end of the it hyear. Or, $(1+r)^n - \sum ai\ (1+r)^{n-i} = 0$, where $ai = Ai/P0$ and *n* years is the time horizon for the project. Since this equation for IRR involves a polynomial in *r* having *n* as its highest power, there will be n solutions for IRR in all, counting multiple roots of the same value as separate solutions. Let us designate these solutions by $x1$, $x2$, ……….,xn. Then $(r - x1)$ $(r - x2)$ ……. $(r - xn) = 0$. Also, we have $r^n + r^{n-1}(n - a1) +$ + ……+ $(1 - [a1 + +a2 + …+ an]) = 0$. Comparing these two equations, we can write down the following identity:

$$(-1)^n x1x2 \ldots\ldots\ldots\ldots xn = 1 - [a1 + a2 + \ldots\ldots.. + an].$$

(I) In case $\sum ai > 1$; the right hand side of this identity is negative. The left hand side of this identity depends on whether n is even or odd. Let us consider these two scenarios, viz. n = even and n = odd, separately.

(a) n = even, then $x1x2 \ldots\ldots\ldots\ldots xn < 0$. So, all the solutions cannot be real and positive. There has to be at least one negative solution because the product of two complex conjugate solutions is necessarily positive. In general, there would be odd number of real and negative solutions. Thus the number of real and positive solutions, which are the only physically acceptable solutions, should also be odd.

(b) n = odd, then $x1x2 \ldots\ldots\ldots\ldots xn > 0$. Since the product of all the roots is positive, the number of real negative roots can be only even. Hence, the number of real positive roots can be only odd.

In other words, irrespective of whether n is even or odd, the number of physically acceptable solutions is odd if $\sum ai > 1$. Therefore, absence of any real positive solution is absolutely ruled out. The generalized condition for existence of IRR is thus $\sum ai > 1$.

(II) In case $\sum ai < 1$; If the algebraic sum of all future cash flows is less than the magnitude of the initial outflow (not quite an acceptable investment scenario for practicing managers), the number of real, positive solutions is either even or zero irrespective of whether n is even or odd. Therefore, such a scenario cannot but lead to either non- existence of IRR or the existence of two (multiple) IRRs. As a matter of fact, each and every example of an investment proposal cited for absence of IRR and for presence of 2 IRRs involves cash flow patterns that satisfy the condition $\sum ai < 1$. However, the practicing managers may not like to examine such scenarios. If the sum of all future cash flows, even without any discounting, does not come up to the level of the initial cash outflow associated with an investment project, such projects are unlikely to evoke much interest in them.

In one such case of non- existence of IRR, the cash flows from a project are (–) 100 (the initial outflow), followed by 14, 32, 52, 64, 70, (–) 142. Here $\sum ai$<1, and the project is not quite interesting for a practicing manager. However, if the last i.e. the 6th cash flow is adjusted to make $\sum ai$>1, it looks all right and there is no inconsistency associated with IRR.

When the last cash flow is altered to (-) 140, (-) 138, (-) 136, (-) 134, (-) 133, (-) 132.75, there is no acceptable solution for IRR. When the magnitude of this cash flow is further reduced to (-) 132.50, (-) 132.25, (-) 132.10, (-) 132.05, and (-) 132.01, there is still no physically acceptable solution for IRR. When the last cash flow becomes (–) 132 (the cross over value i.e. $\sum ai$ =1.), IRR is zero. This is foreseeable because at this particular value of the last cash flow, $\sum ai$=1 and the product of all the solutions for r is zero. As this cash flow becomes (–) 131.90 or (–) 131.75 or (–) 131.00 or (–) 130.00, the IRR becomes 0.2004, 0.4758, 1.5778 and 2.6797 percent respectively. So, the problem of non existence of any real positive solution for IRR vanishes as soon as $\sum ai$> 1.

(B)Existence of Multiple IRRs[1,2]

Most examples cited for this purpose have 2 IRRs.

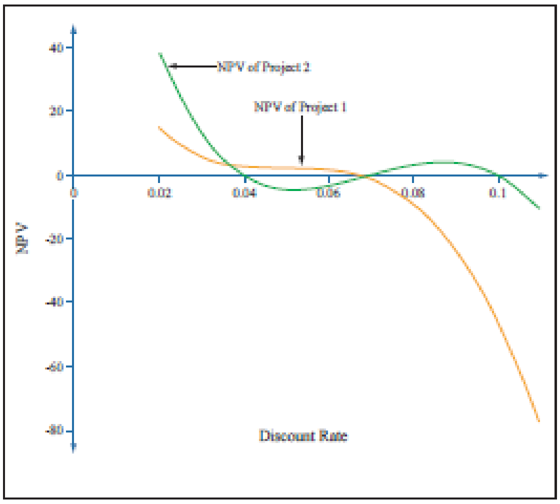
Let us consider the following example cited for the existence of 2 IRRs[2].

The cash flow pattern is (–) 60, 12, ………, 12 (at the end of periods 1 to 9), –15. The IRRs are estimated to be (–) 0.44 and 0.116. As we have already stated earlier, existence of 2 real roots for *r*, one positive and the other negative, does not mean that there are multiple IRRs. It implies existence of a unique IRR. So, there is no problem of multiplicity of IRR in the instant case. Following our method, we can check that $\Sigma\, a\, i$> 1 and hence the existence of a unique IRR is in conformity with our earlier discussions.

Yet another case of existence of two IRRs has a cash flow pattern of (–) 10000, 22500, (–) 12650. The two values of IRR are now 10% and 15%. Here Σai< 1, and the existence of 2 IRRs thus only corroborates our analysis.

In rare cases, however, 3 IRRs can also exist[1].

	CF_0	CF_1	CF_2	CF_3
Project1	–500,000	1,575,000	–1,653,750	578,815
Project2	–500,000	1,605,000	–1,716,900	612,040



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IRR1 = 7%
IRR2 = 4%, 7%, 10%

Under rare circumstances, such a slight adjustment of the cash flow pattern can lead to a sudden switchover from a single IRR to 2 additional IRRs equidistant from the original solution. This is akin to certain situations come across in non-linear dynamics (problems leading to bifurcations[4]). Since such switchovers are expected to be quite rare, practicing managers hardly have any reason to be worried about the same.

(C) Condition for Existence and Uniqueness of IRR:

From the aforesaid analysis, we observe that:

(i) $\sum ai>1$ guarantees existence of IRR, and it is thus the sufficient condition for existence of IRR.

(ii) It is not possible to have a unique IRR without $\sum ai > 1$, and it is thus the necessary condition for uniqueness of IRR.

(iii) Further, if we accept that practicing managers do not get associated with rare, tricky situations involving 3 or more IRRs, $\sum ai > 1$ becomes the necessary and sufficient condition for existence and uniqueness of IRR.

(iv) Practicing managers would restrict their domain of consideration to projects satisfying $\sum ai> 1$. So, they do not face any difficulty in using the IRR method for investment analysis.

(D) Method of application of the Descartes Rule of Signs: In the analysis of IRR, there is a common statement linked with the application of the Descartes Rule of Signs, which states that there are as many internal rates of return for a project as there are changes in the sign of the cash flows[2,5]. Let us try to critically examine this statement to verify whether it is valid or any modification is called for in order to make it precise and correctly applicable.

For this purpose, let us once again focus on some of the cases discussed above.

Case 1: The cash flow pattern is (–) 100, 10, 20, 20, 25, *X*. This is a case where a single outflow of 100 units is followed by five successive annual inflows of 10, 20, 20, 25 and X units. So long as *X* is positive (an inflow), there is only one change in the sign of the cash flows irrespective of the magnitude of *X*. Thus, if the number of IRRs were to be equal to the number of changes in the sign of the cash flows, every positive value of *X* would have ensured the existence of an IRR. But, in reality it is not so. There is an IRR only when $X > 25$. For $X < 25$, no IRR exists. So, this application of Descartes rule of signs is clearly wrong. As we shall see later in this section, there is nothing wrong with the Descartes rule of signs as such. Simply it has not been applied properly.

Case 2: The cash flow pattern is (-) 100, 14, 32, 52, 64, 70, (-) 130. In this case there are two changes in the sign of the cash flows. So, if the number of IRRs were to be related to the number of changes in the sign of the cash flows, there would be either two IRRs or no IRR. However, as we have already seen, there is one real, positive root for IRR in this case. So, this application of the Descartes Rule of Signs is not correct. Once again, this does not mean that there is anything wrong with the Descartes Rule of Signs. We have simply to apply it properly.

In order to identify why the aforesaid way of application of the Descartes Rule of Signs (i.e. linking the number of IRRs to the number of changes in sign of the cash flows) is wrong and how it should be applied properly, let us conduct the following analysis:

Case 1: Let $X = 20$ so that $\Sigma\, ai < 1$. If we write the equation for IRR in decreasing powers of *r*, the coefficients of the terms having r^6, r^5, …… r^0 are respectively 1, 4.90, 9.40, 8.60, 3.35 and 0.05. In other words, there is no change of sign of the terms in decreasing powers of *r*. Consequently, as per the Descartes Rule of Signs, there can be no IRR and this is the position in reality. As a matter of fact, so long as $\Sigma\, ai < 1$, this condition prevails and no IRR exists. However, as $X > 25$ and $\Sigma\, ai > 1$, the sign of the last term turns negative and there is one change of sign in the terms arranged in decreasing powers of r. So, one IRR should exist and in reality it does exist. It is important to note that our findings of the previous section are perfectly consistent with the predictions as per the Descartes Rule of Signs.

Case 2: If we write the equation for IRR in decreasing powers of *r*, the coefficients of the terms having r^6, r^5, …… r^0 are respectively 1, 5.86, 13.98, 16.80, 9.48, 0.48, (-) 0.02. Here the number of change of sign is but one. Hence the existence of only one positive real solution for IRR is in conformity with the Descartes rule of signs. As a matter of fact, so long as $\Sigma\ ai > 1$, this condition prevails and only one IRR exists. However, if $\Sigma\ ai < 1$, there is no change of sign of the terms in decreasing powers of r and no IRR exists as per the Descartes rule of signs. This once again shows that our findings from the previous sections are consistent with those predicted by the Descartes's Rule of Signs.

It is quite clear from the above analysis that the statement relating the number of IRRs to the number of changes of sign of the cash flows is not correct. The number of real, positive solutions for IRR should instead be either equal to the number of changes of sign in the equation in decreasing powers of *r* or less from it by multiples of two. Multiple roots of the same value are to be counted separately for this purpose. This is precisely what the Descartes Rule of Signs states and any deviation from this principle will tantamount to its improper application and is apt to lead to wrong results.

That it is wrong to link the number of IRRs to the number of changes in sign of the cash flows can also be traced to the fact that every real, positive root for [1/(1+*r*)] does not correspond a real, positive root for *r*. Although a real, positive value for *r* implies a real, positive value for [1/(1+*r*)], the converse is obviously not true.

(E) Incorrect project ranking using IRR for mutually exclusive projects[1]:

i) Projects of different scales:

	CF_0	CF_1	IRR	NPV at 10%
Project1	–10,000	20,000	100%	8,181.82
Project2	–20,000	36,000	80%	12727.27

One workaround suggested for this problem is to use the incremental cash flows. If IRR_{2-1} > Discount Rate, project 2 is better than project 1. However, if IRR_{2-1} < Discount Rate, project 1 is better than project 2.

ii) Projects with identical outlays but different time patterns of cash flows:

We consider the following example:

Project 1: $Po = 100$, $A1 = X1$, Ai for $i \neq 1$ is zero, the rate of discount being 10%

Project 2: $Po = 100$, $An = Xn (n > 1)$, Ai for $I \neq n$ is zero, the discount rate being 10%

For project 1, $(NPV)1 = [X1 / 1.1] - 100$,

and $(1 + r1) = X1 / 100$, where $r1$ is the IRR.

For project 2, $(NPV)2 = [Xn / (1.1)^n] - 100$, $[n > 1]$

and $(1 + r2)^n = Xn / 100$, where $r2$ is the IRR.

Thus, $Xn / X1 = [(1+r2)^n / (1+r1)]$

and $(NPV)1 - (NPV)2 = [X1/1.1] - [Xn/(1.1)^n]$

$= X1/(1+r1) [\{(1+r1) / 1.1\} - \{(1+ r1) * Xn\} / \{X1 * (1.1)^n\}]$

$= 100 [\{(1+r1) / 1.1\} - \{(1+r2)^n / (1.1)^n\}]$.

Although $r1$ may be larger than $r2$, it is possible that $(NPV)1 < (NPV)2$ because of the large number of iterations arising from a large value of $n (r1 > r2 > 0.1)$. In such a situation, $r1 > r2$ and $(NPV)1 < (NPV)2$. There may be thus a whole range of situations where the NPV and IRR methods lead to contradictory results.

The above position can be clarified by a rather simple illustration as under: –Let $r1 = 0.15$ (15% p.a.), and $r2 = 0.13$ (13% p.a.); $r1 > r2 > 10\%$

Then $[(1+ r1)/1.1] = 1.04545$, $[(1+r2)/1.1] = 1.018018$; and $[(1+r2)/1.1]^3 = 1.055034 > (1+r1)/1.1$

Thus, although $r1 > r2$, $[(1+r2)/1.1]^3 > [(1+r1)/1.1]$. Thus, $(NPV)1 < (NPV)2$ for $n = 3$ and above. For different values of $r1$ and $r2$, different values of n may be required for (NPV)2 to exceed (NPV)1. This is possible only because n can be made requisitely large by sufficiently deferring the cash flows for project 2.

But, more distant/deferred are the cash flows, the higher is the associated uncertainty and we need to deal with higher discount rates to take care of this feature ($K_2 > K_1$).

It is not possible to predict whether (NPV)2 is still greater than (NPV)1 with $K_2 > K_1$. However, if $r1 > r2$, and $K_2 > K_1$, then $r1 - K_1 >> r2 - K_2$. So, while a NPV based comparison loses much of its force, the IRR based criterion gets further fortified. In other words, IRR should enjoy an edge over NPV for the purpose of comparison between mutually exclusive projects.

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